TITLE .... Plurigenera of 3-folds and Weighted Hypersurfaces. ....

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INSTITUTION and DATE .... University of WARWICK 1988 ....

Plurigenera of 3-folds and Weighted Hypersurfaces.

by

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Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick.

The Mathematics Institute,
University of Warwick,
Coventry.
August 1988.

Summary

Chapter I gives basic results and definitions for nonsingular varieties, normal varieties and canonical singularities.

In Chapter II we give alternative forms of the Riemann-Roch formula for projective 3-folds with at worst canonical singularities. We show for a canonical 3-fold X with $\chi(\mathcal{O}_X) = 1$ that $P_{12}(X) \geq 1$, $P_{24}(X) \geq 2$ and $K_X^3 \geq (\frac{1}{168})^3$. The last section of Chapter II shows that the record of pluridata representing a canonical 3-fold is unique.

In Chapter III we find necessary and sufficient conditions for weighted complete intersections of codimensions 1 and 2 to be quasismooth. We also give conditions for quasismooth surface and 3-fold intersections of codimension 1 and 2 to have at worst only isolated canonical singularities. We produce lists of such complete intersections in two different ways: one using these conditions for quasismoothness and having only isolated canonical singularities and the second deducing the degrees of the generators and relations from the plurigenera via the Poincaré series of the canonical ring.

Algebraic geometry: the field seems to have aquired the reputation of being esoteric, exclusive and very abstract with adherents who are secretly plotting to take over the rest of mathematics.

David Mumford.

Varieties are the spice of life.

An old Algebraic Geometric proverb.

Contents.

Acknowledgements

I would like to thank my supervisor Dr. Miles Reid for all of his help. I would also like to thank my family, friends and the staff of the mathematics departments of the Universities of Warwick and Leicester.

This work was funded by the Science and Engineering Research Council (studentship no. 83301507) and written up under the University of Leicester.

The main results of Chapter II (Theorems II.4.6 and II.4.7) are based on a suggestion of J. Kollár.

I would also like to thank Y. Kawamata for many useful conversations whilst he was on a SERC funded visit to the Mathematics Institute, University of Warwick during Spring 1985.

Introduction

General synopsis.

In this thesis we discuss the following:

(i) the combinatorics of Reid's *exact plurigenera formula*, with applications to finding examples of canonical and anti-canonical models of 3-folds.

(ii) the combinatorics of weighted complete intersections of codimensions 1 and 2 and various notions of *goodness* for these intersections.

Weighted complete intersections.

Du Val, in 1934, classified isolated rational surface singularities which could be embedded in $\mathbf{A}^3$. These singularities arise in a large number of different ways (see [Du]) and in fact are the only canonical surface singularities. Toric geometrical constructions lends themselves very nicely to the study of canonical singularities (see [Da] for an introduction to toric methods). In particular, isolated hypersurface singularities in $\mathbf{A}^{n+1}$ can be studied via the Newton polyhedron (see Definition III.3.16) of the defining polynomial f. Moreover if f is homogeneous with respect to some set of weights $\{a_0, ..., a_n\}$ then the hypersurface $X : (f = 0) \subset \mathbb{P}(a_0, ..., a_n)$ can also be studied by such techniques. Note that the volume of the interior of the Newton polyhedron is related to the genus of X, and the Fine interior (due to J. Fine) is related to plurigenera. For complete intersections of higher codimension there are corresponding results involving the Minkowski mixed volume (see Definition III.3.17).

The techniques of toric geometry (e.g. cones, fans, polyhedra) are in practice hard to compute for general toric spaces, for example the calculation of the Fine interior (see [R4, section 4.12]). For weighted projective spaces and general weighted hypersurfaces the Newton polyhedra are either simplexes or derived from simplexes and so are easier to work with.

A large quagmire of examples of surfaces and 3-folds with canonical singularities is provided by weighted complete intersections in weighted projective space. These are defined by Dolgachev in [WPS]. S_{10} in $\mathbb{P}(1, 1, 2, 5)$ is a famous example of a nonsingular surface with $\omega_S \cong \mathcal{O}_S(1)$, $p_g = 2$ and $K_S^2 = 1$, due to Enriques and studied by Kodaira and many others.

In 1979 Reid first calculated, via a finite tree search, a list of all K3 weighted hypersurfaces with at worst canonical singularities; of which there are 95. This and other lists of surfaces and 3-folds appear in Chapter III. The lists of canonically and anti-canonically embedded 3-folds in sections III.7 and III.8 were generated by a computer search program. The search was in order of increasing degree (or increasing sum of the degrees in the codimension 2 cases) and is therefore not *a priori* finite. However the examples found were produced relatively quickly using a truncated infinite search, after which no more were forthcoming. A number of the searches were pursued for an extended time but produced no extra examples. So I conjecture that they are complete lists of all well-formed quasismooth canonically and anti-canonically embedded weighted complete intersections in codimensions 1 and 2 with at worst terminal singularities. Further evidence in favour of this conjecture is the list in section III.9, produced via a unrelated method. This table method produced no extra examples in codimensions 1 or 2.

An important concept in Chapter III is the quasismoothness of a complete intersection X (i.e. the smoothness of the affine cone over X outside the origin). Although used by other authors (for example see [Da, section 14.1] and [WPS]) none of them have given combinatoric conditions for a weighted complete intersection to be quasismooth. Necessary and sufficient conditions for both the general hypersurface and the codimension 2 cases are given in section III.3. Both these sets of conditions involve the existence of a sufficient number of distinct monomials in particular degrees.

One surprise was the example $X_{12,14}$ in $\mathbb{P}(2, 3, 4, 5, 6, 7)$ which contradicts the conjecture that $H^0(\mathcal{O}_X(-K_X)) \neq 0$. I hunted for other examples of complete intersections with this behaviour amongst lists of Q-Fano 3-folds with $\omega_X \cong \mathcal{O}_X(\alpha)$ for $\alpha \leq -2$, but found none.

Section III.9 uses a method originally used by Reid to produce examples of K3 surfaces. From the Poincaré series of the graded ring corresponding to a weighted complete intersection the degrees of the generators and relations can be found. This technique uses repeated differencing to evaluate the power series. Using the plurigenera formula from section II.4, a Poincaré series can be produced from a record of pluridata (see Definition II.4.8), which we hope will correspond to a canonically embedded 3-fold weighted complete intersection. Likewise we use the anti-plurigenera formula to produce anti-canonically embedded 3-folds. Clearly there will be a large number of rejected records and hence this method is time consuming.

This process produced all of the lists given in sections III.8 and III.9, along with some extra complete intersections in codimensions 3 and 5.

Classification.

The classification of algebraic curves and surfaces rests on the existence of a unique minimal model in dimensions 1 and 2. However the classification of 3-folds is not so straightforward.

In 1979 Reid introduced the concepts of canonical singularities and canonical models of varieties (see [R1]). These canonical models are candidates for minimal models of 3-folds. Under the assumption that such models exist he developed these concepts in [R2]. [R4] is an excellent introduction to such concepts and [Ko3], [Mo1] and [W2] are survey articles on the birational classification of varieties. Starting from 1978 Mori introduced several new ideas, allowing large steps to be taken toward the proof of the existence of minimal models.

One of the most important differences between the studies of surfaces and of 3-folds is the contraction of 1-dimensional subvarieties as well as those in codimension 1. A class of elementary transformations of 3-folds, now called flops, was used by Kulikov in 1977 to study the birational transformations of smooth 3-folds. These transformations were extended by Reid to the case of terminal singularities and by Kawamata to canonical singularities. Currently Kawamata, Kollár, Mori, Reid and many others are studying other elementary transformations called directed flips.

The existence of a minimal model for 3-folds, unique up to flops, was proved by Mori in 1986. This model is allowed to contain terminal singularities and in fact it now seems natural to allow minimal models to have these singularities.

The exact plurigenera formula.

The Riemann-Roch formula, which connects cohomological invariants with geometrical properties of codimension 1 subvarieties on smooth varieties, is a powerful tool in algebraic geometry. It is thus reasonable to extend it to varieties with canonical singularities and $\mathbf{Q}$-divisors on them, including canonical models. This was done by Reid in 1985 (see [R4]), using the Atiyah-Singer Equivariant Index Theorem. For any canonical 3-fold X (or $\mathbf{Q}$-Fano 3-fold) this Riemann-Roch formula can be used to deduce a formula for the plurigenera (respectively anti-plurigenera) involving terms in K_X^3, $\chi(\mathcal{O}_X)$ and contributions due to the canonical singularities on X. These contributions were first calculated numerically from examples of $\mathbf{Q}$-Fano 3-fold weighted hypersurfaces. However it was not until Reid produced his formula for these contributions (see [R4, Theorem 8.5]) that they were seen to follow any set pattern.

Chapter II of this thesis uses Reid's extension of Riemann-Roch to deduce properties of the plurigenera for 3-folds of general type. For varieties of general type two natural questions to consider are:

(1) for which n are the plurigenera P_n positive?

(2) for which n are the n-canonical maps birational?

Section I.7 gives the well-known answers for curves and surfaces. Matsusaka in 1968 proved that for a canonically polarised 3-fold X of general type, $P_n(X) > 0$ for all $n \geq 3$ and ϕ_{nK_X} is birational for all $n \geq 25$. These questions for 3-folds of general type are answered in a limited sense in section II.4.

The plurigenera formula leads to the following question: given a canonical 3-fold X, are the invariants K_X^3, $\chi(\mathcal{O}_X)$, $p_g(X)$ and its basket of singularities uniquely determined by the complete list of plurigenera $\{P_n(X)\}$? This is answered in the affirmative in section II.7 by purely arithmetic methods. It is not known if there is a more intrinsic proof of this fact. Corresponding statements can be made for the case of $\mathbf{Q}$-Fano 3-folds since the plurigenera formulas are very similar.

The first step in the process of isolating these invariants is to calculate the global index R (see Theorem II.7.3). This step however requires all the plurigenera. Consequently it leaves something to be desired as a practical method of determining the invariants of X. After this step only a finite number of plurigenera are needed, and if some other way of limiting the global index can be found this technique could be used practically.

Given that a canonical model X exists for some 3-fold of general type it is thus theoretically possible to calculate the invariants K_X^3, $\chi(\mathcal{O}_X)$, $p_g(X)$ and its basket of singularities uniquely from the plurigenera of the original 3-fold.

I

Preliminaries.

I.1 Preamble

This chapter contains the basic notation used throughout and some well-known background properties of algebraic varieties of dimension ≤ 3. A solid introduction to algebraic geometry is [Hart].

Section I.6 contains an extension of Kodaira vanishing to canonical 3-folds; an extension which has gone into folk lore without being written down.

I.2 Notation

A variety V is a projective variety of dimension m over an algebraically closed field $\mathbf{k}$ of characteristic zero.

$\mathbf{k}^*$ is the multiplicative group of nonzero elements of $\mathbf{k}$.

$\mathbf{Z}$, $\mathbf{Q}$ are the rings of integers and rational numbers respectively.

$\mathbf{Z}_r$ is the abelian group $\{0, ..., r-1\}$ under addition modulo r.

$\mathbf{Z}_r^*$ is the group of units of $\mathbf{Z}_r$ under multiplication.

$\{a, ..., \hat{b}, ..., c\}$ is a list with the element b omited.

$\lfloor x \rfloor$ denotes the greatest integer $n \leq x \in \mathbf{Q}$.

$\phi(n)$ is Euler's function (i.e. the number of positive integers less than n and coprime to n).

$\mathbf{A}^m$ is affine m-space.

$\mathbf{P}^m$ is projective m-space.

$\mathbf{P}(a_0, ..., a_n)$ is used to denote a weighted projective space of weights $a_0, ..., a_n$. When no ambiguity can arise this will be simply be denoted by $\mathbf{P}$.

V^0 is the nonsingular locus of V.

$\mathbf{k}(V)$ is the function field of V.

$\mathcal{O}_V$ is the sheaf of regular functions on V.

$\Omega_V^1 = \Omega_{V/\mathbf{k}}^1$ is the sheaf of regular 1-forms on V.

$\Omega_V^n = \Lambda^n \Omega_V^1$ is the sheaf of regular n-forms on V.

ω_V is the sheaf of regular canonical differentials on V^0.

K_V is the canonical divisor corresponding to $\omega_V = \mathcal{O}_V(K_V)$.

Let $\mathcal{L}$ be a coherent sheaf on a projective variety V. Then we write

$h^i(\mathcal{L}) = h^i(V, \mathcal{L}) = \dim \mathrm{H}^i(\mathcal{L})$ and

$\chi(\mathcal{L}) = \sum(-1)^i h^i(\mathcal{L})$.

Let D be a Cartier divisor on V. Then

$h^i(D) = h^i(\mathcal{O}_V(D))$ and

$\chi(D) = \chi(\mathcal{O}_V(D))$.

Let $\mathcal{F}$ be a locally free sheaf of finite rank. Then $\mathcal{F}^* = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{O}_V)$ is the dual sheaf of $\mathcal{F}$. If $\mathcal{F}$ is of rank 1 then $\mathcal{F} \otimes \mathcal{F}^* \cong \mathcal{O}_V$.

Let H be a vector space. Then H' is the dual of H.

The geometric genus $p_g(V) = h^0(\omega_V)$.

$P_n = h^0(V, \omega_V^{\otimes n})$ is the n^{th} plurigenus. For negative n these are referred to as the anti-plurigenera.

$\phi_{\mathcal{L}}$ is the rational map corresponding to the sheaf $\mathcal{L}$.

ϕ_{nK_V} is the morphism corresponding to the sheaf $\mathcal{O}_V(nK_V)$.

I.3 Formulas for Nonsingular Varieties.

I.3.1 Riemann-Roch.

For a divisor D on a smooth curve C we have:

$$\chi(D) = \deg D + \chi(\mathcal{O}_C).$$

For a divisor D on a smooth surface S we have:

$$\chi(D) = \frac{D(D - K_S)}{2} + \chi(\mathcal{O}_S).$$

For a divisor D on a smooth 3-fold X we have:

$$\chi(D) = \frac{D(D - K_X)(2D - K_X)}{12} + \frac{D \cdot c_2}{12} + \chi(\mathcal{O}_X).$$

I.3.2 Adjunction Formula.

Let $Y \subset X$ be a smooth codimension 1 subvariety on a smooth variety X. Then

$$\omega_Y \cong (\omega_X \otimes \mathcal{O}_X(Y)) \otimes \mathcal{O}_Y.$$

Equivalently

$$K_Y = (Y + K_X)_{|Y}$$

I.3.3 Serre Duality.

Let V be a projective Cohen-Macaulay m-dimensional scheme. There exists a coherent sheaf ω_V, called a dualizing sheaf for V, such that for every locally free sheaf $\mathcal{F}$ on V there exists natural isomorphisms:

$$\mathrm{H}^i(V, \mathcal{F}) \cong \mathrm{H}^{m-i}(V, \mathcal{F}^* \otimes \omega_V)'$$

(see [Hart, Corollary III.7.7, p 244]). If V is non-singular then $\omega_V = \Omega_V^m$.

I.3.4 Kodaira Vanishing.

Let V be a projective smooth variety of dimension m over $\mathbf{k}$ and $\mathcal{L}$ be a ample invertible sheaf on V. Then

$$\mathrm{H}^i(V, \mathcal{L} \otimes \omega_V) = 0$$

for all $i > 0$, and

$$\mathrm{H}^i(V, \mathcal{L}^*) = 0$$

for all $i < m$ (see [Hart, Remark III.7.15]).

I.4 Normal Varieties.

I.4.1 The Canonical Sheaf.

Let V be a normal variety and $j : V^0 \longrightarrow V$ be the inclusion map. The following definitions are equivalent (see [R1, appendix to section 1]):

I.4.2 Definition:

(i) $\omega_V = j_* \omega_{V^0}$.

(ii) $\omega_V = \{ s \in \Omega^m_{\mathbf{k}(V)/\mathbf{k}} : s \text{ is regular in codimension } 1 \}$.

(iii) $\omega_V = (\Omega_V^m)^{**}$.

I.4.3 Note. If V is nonsingular then these agree with the usual definition of $\omega_V = \Omega_V^m$.

I.4.4 The Canonical Divisor.

ω_V is a divisiorial sheaf (see [R1, appendix to section 1 Theorem 7]) and so there exists a Weil divisor K_V such that $\mathcal{O}_V(K_V) \cong \omega_V$ (see [R1, appendix to section 1 Theorem 4]). This divisor is called the *canonical divisor* of V. This is a $\mathbf{Q}$-divisor and is Cartier outside the singular locus.

I.4.5 The Canonical Model.

Let V be an m-dimensional variety of general type.

I.4.6 Definition: *The canonical ring $R(V)$ of V is:*

$$R(V) = \bigoplus_{n \geq 0} \mathrm{H}^0(V, nK_V).$$

Assuming that $R(V)$ is finitely generated then the canonical model X of V is $X \cong$ Proj $R(V)$.

The canonical model is normal and will have at worst canonical singularities. For a curve the canonical model will be smooth, and for a surface it has at worst Du Val singularities.

I.5 Singularities.

I.5.1 Canonical Singularities.

Let V be a projective variety. V has at worst *canonical* (respectively *terminal*) singularities if

(i) there exists an $r \geq 1$ such that rK_V is Cartier.

(ii) if $f : W \longrightarrow V$ is a resolution and $\{E_i\}$ the exceptional prime divisors then

$$rK_W = f^*(rK_V) + \sum_i a_i E_i$$

where $a_i \geq 0$ (respectively $a_i > 0$).

The least such r is called the *global index* of V (usually written R).

Similarly there is the corresponding local definition of a canonical (respectively terminal) point $P \in V$. The least such r is called the *index* of P in V.

I.5.2 Definition: *Suppose V is a projective variety with at worst canonical singularities. V is a canonical variety if K_V is ample; V is a $\mathbf{Q}$-Fano variety if $-K_V$ is ample.*

I.5.3 Du Val Points and Surface Singularities.

Let S be a surface. A Du Val point $P \in S$ (also called rational double points, Klein singularities, etc.) is characterised by the following 2 conditions:

(i) ω_S is invertible at P,

(ii) if $f : W \longrightarrow S$ is a minimal resolution of P then $f^*\omega_S = \omega_W$.

Clearly, Du Val singularities are surface canonical singularities of index 1. There are many other characterisations of these points (see [Du]). An alternative definition is the following:

$P \in S$ is locally analytically isomorphic to $\mathbf{A}^2 / G$, where G is a finite subgroup of $SL_2(\mathbf{k})$.

In particular the group $G \cong \mathbf{Z}_{n+1}$ acting via:

$$u \longmapsto \varepsilon u$$

$$v \longmapsto \varepsilon^{-1} v$$

where u, v are coordinates on $\mathbf{A}^2$ and ε is an $(n+1)$th root of unity. This gives rise to a singularity of type A_n:

$$A_n: x^2 + y^2 + z^{n+1} = 0$$

in $\mathbf{A}^3$. The other types of Du Val point are:

$$D_n: x^2 + y^2 z + z^{n-1} = 0 \text{ for } n \geq 4,$$
$$E_6: x^2 + y^3 + z^4 = 0$$
$$E_7: x^2 + y^3 + yz^3 = 0$$
$$E_8: x^2 + y^3 + z^5 = 0$$

I.5.4 Canonical 3-fold Singularities.

Terminal 3-fold singularities were classified by Reid, Danilov, Mori, Morrison and Stevens (see [R4, section 6]).

I.5.5 Theorem: *The 3 dimensional terminal singularities are the following:*

(i) *smooth points;*

(ii) *isolated singularities given by an equation of the form $f(x, y, z) + tg(x, y, z, t)$, where f is one of those listed in I.5.3;*

(iii) *a few of the cyclic quotients of cases (i) and (ii), of which there is a complete list* (see [R4, section 6.1]).

In this thesis we will only be considering cyclic quotient singularities.

I.5.6 Definition: *Let r, a_1, ..., a_n be positive integers. Let ε be a primitive rth root of unity acting on* $\mathbf{A}^n$ *via*

$$\varepsilon(x_1, \ldots, x_n) = (\varepsilon^{a_1} x_1, \ldots, \varepsilon^{a_n} x_n).$$

A singularity $Q \in X$ is of type $\frac{1}{r}(a_1, \ldots, a_n)$ if (X, Q) is isomorphic on an analytic neighbourhood to $(\mathbf{A}^n, 0) / \langle\varepsilon\rangle$ (see also [R4, 4.2] for notation).

I.5.7 Note. The type of a singularity is unique up to multiplication of the a_i by a unit of $\mathbf{Z}_r$ and permutation of the coordinates.

We have the following theorem from [R4, section 4.11]:

I.5.8 Theorem: *A cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ is canonical (respectively terminal) if and only if:*

$$\frac{1}{r} \sum_i \overline{k a_i} \geq 1$$

(respectively > 1) for all $k = 1, \ldots, r-1$, where $\overline{k a_i}$ denotes residue modulo r.

By the Terminal Lemma (see [R4, section 5]) the only canonical cyclic quotient surface singularities are of the type $\frac{1}{r}(1,-1)$ for $r \geq 1$. Similarly the only terminal cyclic quotient 3-fold singularities are of the type $\frac{1}{r}(1,-1,b)$ for $r \geq 1$ and b coprime to r.

A singularity of type $\frac{1}{r}(1,-1,b)$ is of index r.

I.5.9 Note. A 3-fold singularity which is locally analytically isomorphic to $\mathbf{A}^1 \times P$, where P is a Du Val surface singularity, is canonical.

I.6 Kodaira Vanishing for Canonical Singularities.

We have the following theorem due to Kawamata and Viehweg (see [W2, Theorem 2.4]).

I.6.1 Theorem: *Let X be a smooth projective variety and D be a nef, big $\mathbf{Q}$-divisor, whose support has at worst simple normal crossings. Then*

$$\mathrm{H}^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$$

for all $i > 0$ where $\lceil D \rceil$ is the round-up of D.

We use this to prove the following version of Kodaira vanishing.

I.6.2 Theorem: *Let Y be a canonical m-fold. Then*

$$\mathrm{H}^i(Y, \mathcal{O}_Y(nK_Y)) = 0$$

for all $i > 0$ and $n \geq 2$.

Proof. Let $f: X \longrightarrow Y$ be a resolution such that f^*K_Y is supported on a divisor with only simple normal crossings. Let $\{E_i\}$ be the exceptional prime divisors. Let $D = (n-1)f^*K_Y$ for some $n \geq 2$. Then D is a nef, big $\mathbf{Q}$-divisor since K_Y is ample. So by the above theorem:

$$\mathrm{H}^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$$

for all $i > 0$ and

$$R^i f_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0$$

for all $i > 0$. Therefore there exists natural isomorphisms

$$\mathrm{H}^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) \cong \mathrm{H}^i(Y, f_* \mathcal{O}_X(K_X + \lceil D \rceil))$$

Define the $\mathbf{Q}$-divisors A and $\mathcal{D}$ by:

$$\lceil D \rceil = (n-1)f^*K_Y + A$$

$$\mathcal{D} = K_X + \lceil D \rceil .$$

Since K_Y is Cartier away from the singular locus, A has support only on the exceptional divisors $\{E_i\}$ of f.

As f is a resolution of a canonical variety

$$K_X = f^* K_Y + \Delta$$

where $\Delta = \sum_i d_i E_i$ and $d_i \in \mathbf{Q}$, $d_i \geq 0$. So

$$\begin{aligned} f_* O_X(\mathcal{D}) &= f_* O_X(f^* K_Y + \Delta + (n-1) f^* K_Y + A) \\ &= f_* O_X(n f^* K_Y + \Delta + A). \end{aligned}$$

The $\mathbf{Q}$-divisor $\Delta + A$ is completely supported on the exceptional locus (i.e. $(\Delta + A)_{|f^{-1}Y^0} = 0$). So

$$\begin{aligned} \Gamma(Y^0, f_* O_X(\mathcal{D})) &= \Gamma(f^{-1} Y^0, O_X(n f^* K_Y + \Delta + A)) \\ &= \Gamma(f^{-1} Y^0, O_X(n f^* K_Y)) \\ &= \Gamma(Y^0, f_* O_X(n f^* K_Y)) \\ &= \Gamma(Y^0, O_Y(n K_Y)) \end{aligned}$$

As the two divisorial sheaves $f_* O_X(\mathcal{D})$ and $O_Y(nK_Y)$ agree on an open set whose compliment has codimension ≥ 2 then they are equal (see [R1, Proposition 2 in the appendix to section 1]) Therefore

$$\mathrm{H}^i(Y, O_Y(nK_Y)) = \mathrm{H}^i(Y, f_* O_X(\mathcal{D})) = 0$$

for all $i > 0$.

$\square$

I.7 The n-canonical maps

Let V be an m-dimensional variety of general type. There are 2 natural questions:

(1) when is ϕ_{nK_V} generically finite?

(2) when is ϕ_{nK_V} birational?

[W1] answers this in the case of canonically polarised 3-folds (i.e. smooth and K_V ample) and gives the following table.

V	$n : P_n(V) > 0$	ϕ_{nK_V} generically finite	ϕ_{nK_V} birational
curves of general type	1	1	1
smooth surfaces of general type	2	3	5
canonically polarised 3-folds	3	8	25

The result for canonically polarised 3-folds was originally proved by Matsusaka. The question for canonical 3-folds X with $\chi(O_X) \leq 1$ is answered in section II.4.

We have the following due to Hanamura (see [Han, Theorem 3.4]):

I.7.1 Theorem: *Let X be a nonsingular 3-fold of general type which has a minimal model of index r. Then ϕ_{nK_X} is birational for all $n \geq N$ where*

$$N = \begin{cases} 9 & \text{if } r = 1 \\ 13 & \text{if } r = 2 \\ 4r+4 & \text{if } 3 \leq r \leq 5 \\ 4r+3 & \text{if } r \geq 6 \end{cases}$$

This approaches the main question in section II.4 by fixing the global index instead of $\chi(\mathcal{O}_X)$.

I.8 Minimal models

I.8.1 Definition: *The canonical ring $R(V, K_V)$ of a variety V is given by:*

$$R(V, K_V) = \bigoplus_{n \geq 0} \mathrm{H}^0(V, nK_V).$$

The Kodaira dimension $\kappa(V)$ of V is $\operatorname{tr} \deg_k R(V, K_V) - 1$.

An alternate definition of Kodaira dimension is that it is the maximum dimension of the images $\phi_{nK_V}(V)$. Note that Kodaira dimension is a birational invariant and that it takes values in the range $-\infty, 0, \ldots, \dim V$. Varieties which have Kodaira dimension equal to their dimension as said to be of *general type*.

We have the following in dimension 1 (see [Hart, Section 6 p. 422]):

I.8.2 Theorem: *Let C be a curve of genus g. Then*

(i) $\kappa(C) = -\infty$ *if and only if* $g = 0$.

(ii) $\kappa(C) = 0$ *if and only if* $g = 1$.

(iii) $\kappa(C) = 1$ *if and only if* $g \geq 2$.

Every birational class has a unique nonsingular projective curve in $\mathbf{P}^3$.

For surfaces (see [B] for a comprehensive study of surfaces) we have:

I.8.3 Theorem: *Let S be a surface. Then*

(i) $\kappa(S) = -\infty$ *if and only if* $P_n(S) = 0$ *for all* $n \geq 1$, *if and only if S is ruled (Enriques' Theorem).*

(ii) $\kappa(S) = 0$ *if and only if* $P_n(S) = 0$ *or* 1 *for all* $n \geq 1$ *and* $P_N = 1$ *for some N.*

(iii) $\kappa(S) = 1$ *if and only if* $P_N(S) \geq 2$ *for some* $N \geq 1$ *and* $\phi_{nK_S}(S)$ *is at most a curve for all* n.

(iv) $\kappa(S) = 2$ *if and only if* $\phi_{NK_S}(S)$ *is a surface for some* N.

For birational equivalence classes with Kodaira dimension $\neq -\infty$ there is a unique minimal nonsingular model. Let S be a surface with $\kappa(S) \neq -\infty$ and let $\pi : \tilde{S} \longrightarrow S$ be a resolution of singularities. By contracting all the (-1)-curves on $\tilde{S}$ via $\sigma : \tilde{S} \longrightarrow \hat{S}$, the minimal model $\hat{S}$ is obtained.

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ S \arrow[rr, dashed] & & \hat{S} \end{tikzcd}$$

It is well known that this is unique and the order of contraction of the (-1)-curves does not matter. For surfaces of general type, contracting the (-2)-curves on $\hat{S}$ gives the canonical model (with possibly Du Val singularities).

For 3-folds the classification is under current study. We have the following result by Mori in dimensions at most 3 (see [Ko3, 13.1]):

I.8.4 Theorem: *Let X be a projective variety with* $\dim X \leq 3$. *Then X is birational to a projective* $\mathbb{Q}$*-factorial variety Y, which has at worst terminal singularities, such that one of the following holds:*

(i) *Y admits a Fano contraction $f : Y \longrightarrow Z$ (i.e. f is the contraction of a negative extremal ray and* $\dim Z < \dim Y$*),*

(ii) *K_Y is nef.*

For a given X only one of these cases occurs.

For 3-folds this Y is not unique, although the types of the terminal singularities, which occur on Y, are unique.

II

Plurigenera of 3-folds.

II.1 Preamble

The aim of this chapter is to show how to calculate $\chi(\mathcal{O}_X(nK_X))$ as a function of some data, which we call a record, of a projective 3-fold X with only canonical singularities. Theorems II.2.2 and II.2.4 give four equivalent formulas for $\chi(\mathcal{O}_X(nK_X))$.

The main application of these results (Theorems II.4.6 and II.4.7) are that $P_{12} \geq 1$ and $P_{24} \geq 2$ for a canonical 3-fold X with $\chi(\mathcal{O}_X) = 1$. This can be compared with surfaces of general type (see also section I.6).

Sections 1 – 5 of this chapter have been published as [F].

Section 6 describes results for the anti-plurigenera of $\mathbf{Q}$-Fano 3-folds. Section 7 solves the 'inverse' problem of knowing how the plurigenera determine the record. However at present this result is ineffective in the sense that it only allows the record to be determined from the infinite list of plurigenera.

II.2 Contributions to Riemann-Roch

Throughout section 2 we assume that X is a projective 3-fold with only canonical singularities.

II.2.1 Definition: *Suppose Q is a singularity of type $\frac{1}{r}(a, -a, 1)$ where r and a are coprime. Then Q is also of type $\frac{1}{r}(1, -1, b)$, where $ba \equiv 1 \bmod r$, and is a terminal singularity (see Theorem I.5.8). Fix a primitive nth root of unity, ε and define*

$$\sigma(Q, n) = \sum_{k=1}^{r-1} \frac{\varepsilon^{nk}}{(1-\varepsilon^{k})(1-\varepsilon^{ak})(1-\varepsilon^{-ak})}.$$

This is a rational number independent of the choice of ε (see [R4, sections 8.7, ..., 8.10]).

By [R4, section 8.6] we have the following:

II.2.2 Theorem: *There exists a list $\mathcal{B}$ of types of 3-fold cyclic quotient singularity such that*

$$(1):\quad \chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12}K_X^3 + \chi(\mathcal{O}_X) + \frac{n\pi^*K_X \cdot c_2(Y)}{12} + \sum_{Q\in\mathcal{B}} \frac{1}{r}(\sigma(Q,n)-\sigma(Q,0)),$$

for all n, where $\pi : Y \longrightarrow X$ is a resolution.

II.2.3 Note. This list $\mathcal{B}$ of types is called a *basket* of singularities and is not necessarily a list of the singularities actually occurring on X. However the singularities of X make the same contribution to $\chi(\mathcal{O}_X(nK_X))$ as those of the basket (see [R4, 8.2]).

For example the 3-fold $\mathbb{P}(1, 2, 3, 4)$ has an isolated singularity of type $\frac{1}{3}(2, 1, 1)$ at $[0, 0, 1, 0]$, a line of singularities of type A_1 along the line $[0, x, 0, z]$ and a non-isolated singularity of type $\frac{1}{4}(1, 2, 3)$ on that line at $[0, 0, 0, 1]$. The basket of singularities representing this 3-fold contains 1 of type $\frac{1}{3}(2, 1, 1)$ and 2 of type $\frac{1}{2}(1, 1, 1)$. See section III.2 for definitions and examples of weighted projective spaces.

This formula can be written in different forms; some more useful than others depending on the application.

II.2.4 Theorem: *For all n*

$$(2):\quad \chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12}K_X^3 + \chi(\mathcal{O}_X) + \frac{n\pi^*K_X \cdot c_2(Y)}{12} + \sum_Q \left[-\frac{(r^2-1)\bar{n}}{12r} + \sum_{k=1}^{\bar{n}-1} \frac{\overline{bk}(r-\overline{bk})}{2r} \right],$$

where $\bar{x}$ denotes the smallest residue modulo r.

$$(3):\quad \chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_Q \left[\frac{r^2-1}{12}\left\lfloor \frac{n}{r} \right\rfloor + \sum_{k=1}^{\bar{n}-1} \frac{\overline{bk}(r-\overline{bk})}{2r} \right],$$

where $\lfloor x \rfloor$ denotes the integral part of x.

$$(4):\quad \chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_Q \sum_{k=1}^{n-1} \frac{\overline{bk}(r-\overline{bk})}{2r}.$$

Proof. By [R4, section 8.10]

$$\sigma(Q,n)=\sum_{k=1}^{\bar{n}-1}\overline{bk}(r-\overline{bk})+\frac{r^2-1}{24}(1-2\bar{n})$$

where the sum is taken to be zero if $n\equiv 0$ or 1 mod r. Thus

$$\sigma(Q,n)-\sigma(Q,0)=\sum_{k=1}^{\bar{n}-1}\overline{bk}(r-\overline{bk})-\frac{r^2-1}{12}\bar{n}.$$

So (2) follows from (1).

By an unpublished result of R. Barlow (see also [Ka, section 2] or [R4, Corollary 10.3])

$$\pi^*K_X\cdot c_2(Y)=\sum_Q\frac{r^2-1}{r}-24\chi(\mathcal{O}_X)$$

This gives (3).

(4) is obtained by noticing that

$$\sum_{k=1}^{r-1}\overline{bk}(r-\overline{bk})=\sum_{k=1}^{r-1}k(r-k)=\frac{r(r^2-1)}{6}$$

since r and b are coprime.

□

II.2.5 Definition: *Define*

$$l(Q,n)=\sum_{k=1}^{n-1}\frac{\overline{bk}(r-\overline{bk})}{2r}$$

and $l(n)=\sum_Q l(Q,n)$. This is the correction term $l(n)$ defined in [R1, Theorem 5.5].

II.2.6 Note. There exists a closed formula for $l(Q,n)$ for a singularity Q of type $\frac{1}{r}(1,-1,1)$:

$$l(Q,n)=\frac{\bar{n}(\bar{n}-1)(3r+1-2\bar{n})}{12r}+\frac{r^2-1}{12}\left\lfloor\frac{n}{r}\right\rfloor.$$

This does not happen for other types of singularity (i.e. with $b\neq 1$), since b does not generate $\mathbf{Z}_r$ in a 'linear manner'. Suppose that $b(\bar{n}-1)<r$, then $\overline{bk}=bk$ for $k=1,2,\ldots,\bar{n}-1$. So

$$\begin{aligned}
l(Q,n)&=\sum_{k=1}^{\bar{n}-1}\frac{\overline{bk}(r-\overline{bk})}{2r}+\frac{r^2-1}{12}\left\lfloor\frac{n}{r}\right\rfloor\\
&=\frac{b}{2}\sum_{k=1}^{\bar{n}-1}k-\frac{b^2}{2r}\sum_{k=1}^{\bar{n}-1}k^2+\frac{r^2-1}{12}\left\lfloor\frac{n}{r}\right\rfloor\\
&=\frac{b\bar{n}(\bar{n}-1)}{4}-\frac{b^2\bar{n}(\bar{n}-1)(2\bar{n}-1)}{12r}+\frac{r^2-1}{12}\left\lfloor\frac{n}{r}\right\rfloor\\
&=\frac{b\bar{n}(\bar{n}-1)(3r+b-2b\bar{n})}{12r}+\frac{r^2-1}{12}\left\lfloor\frac{n}{r}\right\rfloor.
\end{aligned}$$

for $\bar{n} < (r/b + 1)$. This reduces to the closed formula for $b = 1$.

II.2.7 Corollary: *The correction term, $l(n)$ is non-negative for $n \geq 2$, and $l(0) = 0 = l(1)$. Moreover if X has at least one singularity of index $\neq 1$ then $l(n)$ is strictly monotonic increasing for all $n \geq 1$.*

Proof. The correction $l(n)$ is formally equal to a sum of $l(Q, n)$ for some basket of singularities Q, each summand being strictly positive for $n \geq 2$.

□

II.2.8 Note. By use of Serre duality and version (4) of the formula, $l(n)$ can be extended for negative n by:

$$l(-n) = -\,l(n+1)$$

for all $n \geq 1$. Compare [R4, exercise 8.11].

II.3 Technical Lemmas

The following lemmas are used in the next section.

II.3.1 Lemma: *For all $m \geq 0$ and $n \geq 1$*

$$l(m+2n) \geq l(m) + nl(2)$$

with equality if and only if all the singularities are of type $\frac{1}{2}(1, 1, 1)$.

Proof. It is enough to prove this for a single singularity Q of type $\dfrac{1}{r}(a, -a, 1)$. Let $ba \equiv 1$ mod r and define

$$\delta_j = \overline{jb}(r - \overline{jb}) + \overline{(j+1)b}(r - \overline{(j+1)b}) - b(r-b).$$

So $2r(l(Q, 2n) - l(Q, m) - nl(Q, 2))$

$$= \sum_{k=m}^{m+2n-1} \overline{bk}(r - \overline{bk}) - nb(r-b)$$

$$= \sum_{j=0}^{n-1} \left[\overline{(2j+m)b}\,(r - \overline{(2j+m)b}) + \overline{(2j+m+1)b}\,(r - \overline{(2j+m+1)b}) - b(r-b) \right];$$

separating even and odd k

$$= \sum_{j=0}^{n-1} \delta_{2j+m}$$

Consider the individual δ_j, and let $\alpha = \overline{jb}$. There are 2 cases to consider:

(i) $\alpha + b < r$. Then $\overline{(j+1)b} = \alpha + b$ and so $\delta_j = 2\alpha(r - \alpha - b) > 0$.

(ii) $\alpha + b \geq r$. Then $\overline{(j+1)b} = \alpha + b - r$ and so $\delta_j = 2(r-\alpha)(\alpha + b - r) \geq 0$.

Thus $\delta_j \geq 0$ for all $j \geq 0$.

□

II.3.2 Lemma: *Suppose $\alpha > \beta$ are integers. Then*

$$l(\frac{1}{\alpha}(1,-1,1),n) \geq l(\frac{1}{\beta}(1,-1,1),n)$$

for all $n < \beta < \alpha$.

Proof. This comes straight from Note II.2.6.

$\square$

II.3.3 Lemma:

$$l(\frac{1}{r}(a,-a,1),n) \geq l(\frac{1}{r}(1,-1,1),n)$$

for all $n \leq \left\lfloor \frac{r+1}{2} \right\rfloor$.

Proof. For any a, $l(\frac{1}{r}(a,-a,1),n)$ is a sum of terms taken from the list $\frac{r-1}{2r}$, $\frac{2(r-2)}{2r}$, $\frac{3(r-3)}{2r}$, ..., $\frac{r-1}{2r}$; each term occurring at most twice. Suppose $\frac{t(r-t)}{2r}$ is such a summand occurring in $l(\frac{1}{r}(a,-a,1),n)$. Then there is an integer k such that $1<k<n-1$ and either $\overline{bk}=t$ or $\overline{bk}=r-t$. So either $k=r-\overline{at}$ or $k=\overline{at}$. But only one of these solutions satisfies $k<n\leq\left\lfloor \frac{r-1}{2} \right\rfloor$, and so each of the summands $\frac{r-1}{2r}$, $\frac{2(r-2)}{2r}$, ..., $\frac{u(r-u)}{2r}$, where $u=\left\lfloor \frac{r-1}{2} \right\rfloor$, occurs only once in $l(\frac{1}{r}(a,-a,1),n)$. Since

$$\frac{r-1}{2r} < \frac{2(r-2)}{2r} < \ldots < \frac{u(r-u)}{2r},$$

then

$$l(\frac{1}{r}(a,-a,1),n) \geq \sum_{k=1}^{n-1} \frac{k(r-k)}{2r} = l(\frac{1}{r}(1,-1,1),n).$$

$\square$

II.3.4 Corollary: *For all α, $\beta \in \mathbb{Z}$ with $0 \leq \beta \leq \alpha$ and for all $n \leq \left\lfloor \frac{\alpha+1}{2} \right\rfloor$, we have*

$$l(\frac{1}{\alpha}(a,-a,1),n) \geq l(\frac{1}{\beta}(1,-1,1),n).$$

II.3.5 Definition: *Define $\Delta_n(Q) = n^2 l(Q,2) + l(Q,n) - l(Q,n+1)$ and $\Delta_n = \sum_Q \Delta_n(Q)$.* The purpose of this definition will become clear in Lemma II.4.10 and Corollary II.4.11.

II.3.6 Lemma: *$\Delta_n(Q)$ is an integer for all types of singularity Q.*

Proof. Clearly $\Delta_n(Q) = (n^2 b(r-b) - \overline{nb}(r-\overline{nb}))\,/\,2r$. Let $x = \left\lfloor \frac{nb}{r} \right\rfloor$. Then

$$\Delta_n(Q) = \frac{x(x-1)}{2}\,r + \frac{n(n+1-2x)}{2}\,b$$
$$= \binom{x}{2} r + \left[\binom{n}{2} - xn\right] b$$

This is an integer. Notice that this particular value of x minimizes the above expression.

□

II.3.7 Note.

$\Delta_2(Q) = \min(r-b, b)$,
$\Delta_3(Q) = \min(3r-3b, r, 3b)$,
$\Delta_4(Q) = \min(6r-6b, 3r-2b, r+2b, 6b)$,
$\Delta_5(Q) = \min(10r-10b, 6r-5b, 3r, r+5b, 10b)$, etc..

The following two lemmas will be of use in section II.7

II.3.8 Lemma: *Consider a periodic function $f: \mathbb{Z} \longrightarrow \mathbb{Q}$ with exact period r. Consider the differenced function $\delta f(n) = f(n+1) - f(n)$. Then δf is periodic with exact period r.*

Proof. $\delta f(n+r) = f(n+r+1) - f(n+r) = \delta f(n)$. So δf is periodic, with period dividing r. Conversely suppose δf is periodic with period s.

$$f(n) = f(0) + \sum_{m=0}^{n-1} \delta f(m)$$

for $n \geq 0$. Thus

$$f(n+s) = f(0) + \sum_{m=0}^{n+s-1} \delta f(m)$$
$$= f(n) + \sum_{m=n}^{n+s-1} \delta f(m)$$
$$= f(n) + \sum_{m=0}^{s-1} \delta f(m)$$

However $f(r) = f(0)$ and so $\sum_{m=0}^{r-1} \delta f(m) = 0$. Hence $\frac{r}{s} \sum_{m=0}^{s-1} \delta f(m) = 0$ and so $\sum_{m=0}^{s-1} \delta f(m) = 0$. Thus $f(n+s) = f(n)$ for all $n \geq 0$. Therefore $r = s$.

□

II.3.9 Lemma: *Let $\mathcal{B}$ be a basket of isolated terminal 3-fold singularities. Then $\delta^3 l(n) = l(n+3) - 3l(n+2) + 3l(n+1) - l(n)$ for this basket has exact period R, the global period of $\mathcal{B}$.*

Proof. Now

$$\delta l(n) = \sum_{Q \in \mathcal{B}} \frac{\overline{nb_Q}(r_Q - \overline{nb_Q})}{2r_Q}$$

has exact period $R = \operatorname{lcm}_{Q \in \mathcal{B}}(r_Q)$. Thus, by the above lemma, $\delta^3 l(n)$ has exact period R.

□

II.4 Non-zero Plurigenera for canonical 3-folds

The main question in this section is to find an integer n such that $P_n(X) \geq 1$ (or $P_n(X) \geq 2$) for every canonical 3-fold X. When $p_g(X) \geq 2$ this question is easily answered (i.e. $P_n \geq 2$ for all $n \geq 1$). As

$$\chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - p_g(X)$$

then $\chi(\mathcal{O}_X) \geq 2$ implies that $h^2(\mathcal{O}_X) \geq 1$; for which it is hoped that other results are possible.

If $h^2(\mathcal{O}_X) = 0$ then $\chi(\mathcal{O}_X) \leq 1$, which is the case dealt with in this section. This is an important situation since $h^2(\mathcal{O}_X) = 0$ whenever X is a weighted projective complete intersection (see Chapter III).

In this section assume that X is a canonical 3-fold, i.e. a projective 3-fold with only canonical singularities and K_X ample. The main results of this section are Theorems II.4.6 and II.4.7. For a different approach see [Han].

Standard use of vanishing (see Theorem I.6.2) gives

II.4.1 Theorem: *For all $n \geq 2$*

$$P_n = h^0(\mathcal{O}_X(nK_X)) = \chi(\mathcal{O}_X(nK_X))$$

Thus for all $n \geq 2$

$$(*) \quad P_n = \frac{(2n-1)n(n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X) + l(n)$$

where $l(n)$ is defined in Definition II.2.5. Notice that the term involving $\chi(\mathcal{O}_X)$ is the only one which can be negative.

II.4.2 Theorem: *If $\chi(\mathcal{O}_X) = 0$ then $P_n \geq 1$ for all $n \geq 2$ and $P_n \geq 3$ for all $n \geq 4$. If $\chi(\mathcal{O}_X) < 0$ then $P_n \geq 4$ for all $n \geq 2$.*

Proof. When $\chi(\mathcal{O}_X) = 0$ then it is clear that $P_n \geq 1$ for all $n \geq 2$. We have

$$P_n \geq P_4 = 2P_2 + 6K_X^3 + l(4) - 2l(2)$$

for all $n \geq 4$. By Lemma II.3.1 $l(4) \geq 2l(2)$ and so $P_{2n} \geq P_4 \geq 3$ for all $n \geq 4$.

□

We have the following due to Kollár [Ko1, Corollary 4.8]:

II.4.3 Theorem: *Let X be a 3-fold of general type. If $P_k(X) \geq 2$ then the $(7k+3)$-canonical map is generically finite and the $(11k+5)$-canonical map is birational.*

The above theorems give:

II.4.4 Corollary: *When $\chi(\mathcal{O}_X) = 0$, the 31-canonical map is generically finite and the 49-canonical map is birational. When $\chi(\mathcal{O}_X) < 0$ the 17-canonical map is generically finite and the 27-canonical map is birational.*

Reid pointed out the following:

II.4.5 Lemma: *Let Z be a canonical m-fold and suppose that the n-canonical map is generically finite. Then $(nK_Z)^m \geq 1$.*

Proof. Let $f : Y \longrightarrow Z$ be a resolution of the base locus of $|nK_Z|$. Define $L = f^* nK_Z$. Then L is a nef $\mathbf{Q}$-divisor and

$$L = M + F$$

where $|M|$ is a free linear system and F is the fixed part of L. Notice that f includes the fractional part of L and is effective. Also M is nef. So

$$\begin{aligned} L^a M^{m-a} &= L^{a-1}(M+L)M^{m-a} \\ &= L^{a-1}M^{m-a+1} + L^{a-1}M^{m-a}F \\ &\geq L^{a-1}M^{m-a+1} \end{aligned}$$

since F is effective. Induction gives $L^m \geq M^m$. As $\phi_M : Z \longrightarrow \mathbf{P}^N$ is generically finite,

$$(nK_Z)^m = L^m \geq M^m = \deg \phi_M \cdot \deg \phi_L(Z).$$

Since $\deg \phi_M \geq 1$ and $\deg \phi_M(Z) \geq 1$, then $(nK_Z)^m \geq 1$.

$\square$

So if $\chi(\mathcal{O}_X) = 0$, then $K_X^3 \geq (\frac{1}{31})^3$ and if $\chi(\mathcal{O}_X) < 0$ then $K_X^3 \geq (\frac{1}{17})^3$.

The rest of this section attempts to generalise this type of result to other values of $\chi(\mathcal{O}_X)$. Kollár pointed out that for any $\chi = \chi(\mathcal{O}_X)$ there is an $n(\chi)$ such that $P_{n(\chi)} \geq 1$, but his values for $n(\chi)$ were huge. We shall calculate a reasonable (and perhaps best possible) bound for $\chi = 1$.

II.4.6 Theorem: *If $\chi(\mathcal{O}_X) = 1$ then $P_{12} \geq 1$.*

II.4.7 Theorem: *If $\chi(\mathcal{O}_X) = 1$ then $P_{24} \geq 2$.*

By II.4.3 it follows that for $\chi(\mathcal{O}_X) = 1$ the 168-canonical map is generically finite and the 269-canonical map is birational. Thus $K_X^3 \geq (\frac{1}{168})^3$. These two theorems will be proved after some preliminaries.

II.4.8 Definition: *A record (or a formal record of pluridata) X is a collection $K^3 \in \mathbb{Q}$, $\chi \in \mathbb{Z}$, $p_g \in \mathbb{Z} \geq 0$, and a basket $\{Q\}$ of singularities. The plurigenera $\{P_n\}$ of X are given by (*). The pluridata of X is the data K_X^3, $\chi(\mathcal{O}_X)$, p_g, and its basket $\{Q\}$ of singularities (see Note II.2.3).*

II.4.9 Example. Consider the formal record of pluridata X; $K^3 = \frac{1}{420}$; $\chi = 1$; $p_g = 0$; and singularities: 2 of type $\frac{1}{2}(1,1,1)$, 2 of type $\frac{1}{3}(2,1,1)$, and one each of types $\frac{1}{4}(3,1,1)$, $\frac{1}{5}(3,2,1)$, and $\frac{1}{7}(5,2,1)$. In this case the plurigenera are:

n	P_n	n	P_n	n	P_n
1	0	9	0	17	1
2	0	10	0	18	2
3	0	11	0	19	2
4	0	12	1	20	3
5	0	13	0	21	3
6	0	14	1	22	3
7	0	15	1	23	4
8	0	16	1	24	5

It is an interesting open question to know if there exists a canonical 3-fold with this record.

II.4.10 Lemma: *For all formal records of pluridata (not necessarily coming from a 3-fold X) the following are equivalent:*

(i) P_2 *is an integer*

(ii) P_n *is an integer for* $n \geq 2$

(iii) $K^3 \equiv -2I(2) \bmod 2\mathbb{Z}$.

Proof. By differencing (*) and using (*) with $n = 2$ to eliminate K_X^3, we get;

$$P_{n+1} - P_n = n^2 P_2 + (3n^2-2)\chi(\mathcal{O}_X) - \Delta_n$$

for all $n \geq 2$ and where Δ_n is given in Definition II.3.5. (ii) follows from (i) by induction and Lemma II.3.6. (i) and (iii) are clearly equivalent.

□

These difference equations give rise to 4 equalities which will be used in the proof of Theorem II.4.6.

II.4.11 Corollary:

(1) $P_3 - 5P_2 = 10\chi - \sum_Q \Delta_2(Q)$.

(2) $P_4 - P_3 - 9P_2 = 25\chi - \sum_Q \Delta_3(Q)$.

(3) $P_6 - P_4 - 41P_2 = 119\chi - \sum_Q (\Delta_4(Q) + \Delta_5(Q))$.

(4) $P_{12} - P_6 - 451P_2 = 1341\chi - \sum_Q (\Delta_6(Q) + \dots + \Delta_{11}(Q))$.

(5) $P_{24} - P_{12} - 3818P_2 = 11430\chi - \sum_Q (\Delta_{12}(Q) + \dots + \Delta_{23}(Q))$.

So the condition $P_2 = 0 = P_3$ limits the number of singularities present.

Proof. Substituting $n = 2$ in the proof of the previous lemma

$$P_3 - P_2 - 4P_2 = 10\chi - \sum_Q \Delta_2(Q).$$

Similarly for (2), (3) and (4). Suppose that $P_2 = 0$ and $P_3 = 0$. Then (1) gives $10\chi = \sum_Q \Delta_2(Q)$. As $\Delta_2(Q) \geq 1$ for each type of singularity, then there are at most 10χ singularities present.

□

II.4.12 Lemma: *Suppose the pluridata X contains a singularity of index r where:*

$$r \geq s = \frac{(12\chi-1)(24\chi-1)}{2(6\chi-1)}.$$

Then $P_{12\chi} \geq 1$.

Proof. Suppose there is a singularity Q present of index $r \geq s$. Let Q' be a singularity of type $\frac{1}{s}(1, -1, 1)$. Then

$$\begin{aligned} l(12\chi) &\geq l(Q, 12\chi) \\ &\geq l(Q', 12\chi) \text{ by Corollary II.3.4,} \\ &= \frac{12\chi(12\chi-1)(3s+1-24\chi)}{12s} \\ &= (24\chi-1)\chi. \end{aligned}$$

So $P_{12\chi} \geq 1$.

□

The above corollary and lemma allow an explicit n_χ to be calculated such that $P_{n_\chi} \geq 1$ for $\chi = \chi(O_X)$.

II.4.13 Proof of Theorem II.4.6. Suppose $\chi(O_X) = 1$ and let the pluridata of X be $\chi = 1$, K_X^3, $(S_i)_{i=0,\dots,m}$. In fact we will prove that P_{12} is non-zero for any pluridata X with $\chi = 1$ and $K^3 > 0$. So if this record of pluridata corresponds to a canonical 3-fold X, then $P_{12}(X) \geq 1$.

Suppose that $\chi = 1$, $K^3 > 0$, $(S_i)_{i=0,\dots,n}$ is a record of pluridata such that $P_{12} = 0$. Since P_{12} is zero, so are p_g, P_2, P_3, P_4, and P_6, and hence this fixes K^3 and limits $l(n)$ by $K^3 = 2(3-l(2)) > 0$. Define $\Gamma_1(S_i) = \Delta_2(S_i)$, $\Gamma_2(S_i) = \Delta_3(S_i)$, $\Gamma_3(S_i) = \Delta_4(S_i) + \Delta_5(S_i)$, and $\Gamma_4(S_i) = \Delta_6(S_i) + \dots + \Delta_{11}(S_i)$, for all $i = 0, \dots, n$. By Corollary II.4.11

(1) $\sum_{i=0}^{n} \Gamma_1(S_i) = 10,$

(2) $\sum_{i=0}^{n} \Gamma_2(S_i) = 25,$

(3) $\sum_{i=0}^{n} \Gamma_3(S_i) = 119,$

(4) $\sum_{i=0}^{n} \Gamma_4(S_i) = 1341.$

As $\Gamma_1(S) \geq 1$ for any singularity S, then there are at most 10 singularities present in the pluridata. By Lemma II.4.12, any singularity appearing in the pluridata must have index less than 26. Hence there are only a finite number of possible combinations of singularities for this pluridata. Appendix II.5 lists the 100 singularities of index less than 26 with the corresponding values of Γ_1, Γ_2, Γ_3, and Γ_4 for each singularity.

Using the ordering of types of singularities in Appendix II.5, let n_j be the number of singularities of the jth type S_j and $\Gamma_{i,j} = \Gamma_i(S_j)$. Let Γ be the 100×4 matrix $(\Gamma_{i,j})$. Then the 4 equations are given by

$$(n_1, \ldots, n_{100})\Gamma = (10, 25, 119, 1341).$$

Column reducing Γ via the matrix E

$$E = \begin{pmatrix} 3 & -2 & -4 & -10 \\ -1 & 1 & -3 & -16 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives

$$(n_1, \ldots, n_{100})\Gamma' = (5, 5, 4, 8).$$

where $\Gamma' = \Gamma E$. The matrix $\Gamma' = (\Gamma_{i,j}')$ is given as the last 4 columns of appendix II.5. This gives 4 new equations:

(5) $\sum_{i=0}^{n} \Gamma_1'(S_i) = 5,$

(6) $\sum_{i=0}^{n} \Gamma_2'(S_i) = 5,$

(7) $\sum_{i=0}^{n} \Gamma_3'(S_i) = 4,$

(8) $\sum_{i=0}^{n} \Gamma_4'(S_i) = 8.$

By reference to Appendix II.5, there are types of singularity which have $\Gamma_4' > 8$. These singularities can never satisfy (8) and can be deleted from the list. Likewise for those singularities with $\Gamma_1' > 5$, $\Gamma_2' > 5$, or $\Gamma_3' > 4$. This reduces the number of types of singularity to just 36.

Suppose there is a singularity S_0 present with $\Gamma_4' = 8$. Then there are 2 cases;

(i) $\Gamma_3'(S_0) = 2$. Thus 2 singularities S_1, S_2 of type $\frac{1}{5}(3, 2, 1)$ are required to satisfy (3). So $\Gamma_2'(S_0) \leq 3$ and $\Gamma_1'(S_0) \leq 3$. So S_0 is of type $\frac{1}{10}(7, 3, 1)$ and a further 3 singularities of type $\frac{1}{2}(1, 1, 1)$ are required.

(ii) $\Gamma_3'(S_0) = 3$. Then S_0 is of type $\frac{1}{5}(4, 1, 1)$. So a singularity of type $\frac{1}{5}(3, 2, 1)$, 3 of type $\frac{1}{3}(2, 1, 1)$ and 4 of type $\frac{1}{2}(1, 1, 1)$ are required to satisfy the equations.

Suppose that $\Gamma_4'(S_0) = 7$. Now there are no types of singularity with $\Gamma_4' = 1$ and so (8) can not be satisfied. So any singularity S with $\Gamma_4'(S) \geq 7$ can be deleted from the list.

By considering each value of Γ_4' in decreasing order, exactly 2 more solutions are found;

(iii) 2 of type $\frac{1}{4}(3, 1, 1)$, 3 of type $\frac{1}{3}(2, 1, 1)$, and 5 of type $\frac{1}{2}(1, 1, 1)$;

(iv) 1 of type $\frac{1}{4}(3, 1, 1)$, 2 of type $\frac{1}{8}(5, 3, 1)$, and 3 of type $\frac{1}{2}(1, 1, 1)$.

In all 4 solutions (i) - (iv) $l(2) = 3$ and so $K^3 = 0$, a contradiction. So there are no pluridata with $\chi = 1$, $K^3 > 0$ and $P_{12} = 0$. This proves Theorem II.4.6.

□

II.4.14 Proof of Theorem II.4.7. Consider a record with $\chi = 1$ and $K^3 > 0$. As $P_{12} \geq 1$ then $P_{24} \geq 1$. Assume that $P_{24} = 1$. Thus P_2, P_3, P_4, and P_6 are either 1 or 0. From the plurigenera formula (*) we have

$$\begin{aligned} P_4 &= 2P_2 + 6K^3 - 1 + l(4) - 2l(2) \\ &> 2P_2 - 1. \end{aligned}$$

So if $P_2 = 1$ then $P_{24} > 1$. Hence we can assume that $P_2 = 0$. There are 6 cases:

(i) $P_6 = 0$, $P_4 = 0$, and $P_3 = 0$;

(ii) $P_6 = 0$, $P_4 = 1$, and $P_3 = 0$;

(iii) $P_6 = 1$, $P_4 = 0$, and $P_3 = 0$;

(iv) $P_6 = 1$, $P_4 = 1$, and $P_3 = 0$;

(v) $P_6 = 1$, $P_4 = 0$, and $P_3 = 1$;

(vi) $P_6 = 1$, $P_4 = 1$, and $P_3 = 1$.

Using Corollary II.4.11, these give 4 equations for each case.

Also $P_{24} = 1$ and so $l(24) \leq 48$. Hence any singularity occurring in the pluridata has index less than 25.

Using the same techniques as in the proof of Theorem II.4.6, the only pluridata with $P_{24} = 1$ are:

(i) 1 of type $\frac{1}{12}(7, 5, 1)$, 1 of type $\frac{1}{4}(3, 1, 1)$, 2 of type $\frac{1}{3}(2, 1, 1)$ and 2 of type $\frac{1}{2}(1, 1, 1)$;

(ii) 4 of type $\frac{1}{6}(5, 1, 1)$ and 6 of type $\frac{1}{2}(1, 1, 1)$;

(iii) 1 of type $\frac{1}{6}(5, 1, 1)$, 4 of type $\frac{1}{3}(2, 1, 1)$ and 5 of type $\frac{1}{2}(1, 1, 1)$.

These solutions occur in cases (i), (ii), and (iii) respectively. Each of these solutions has $K^3 = 0$.

□

II.4.15 Theorem: *If* $\chi(\mathcal{O}_X) = 2$ *then* $P_{24} \geq 1$.

Proof. Using the notation of II.4.13, assume that $P_{24} = 0$ for some record X. So p_g, P_2, P_3, P_4, P_6, P_8 and P_{12} are all zero. By Corollary II.4.11

(1) $\sum_{i=0}^{n} \Gamma_1(S_i) = 20$,

(2) $\sum_{i=0}^{n} \Gamma_2(S_i) = 50,$

(3) $\sum_{i=0}^{n} \Gamma_3(S_i) = 238,$

(4) $\sum_{i=0}^{n} \Gamma_4(S_i) = 2682,$

(5) $\sum_{i=0}^{n} \Gamma_5(S_i) = 22860,$

where $\Gamma_5(S_j) = \Delta_{12}(S_j) + \ldots + \Delta_{23}(S_j)$.

The index of each singularity present in the record X is limited by Lemma II.4.12,

$$r < \frac{(24-1)(48-1)}{2(12-1)} < 50.$$

Repeating the elimination process (as for II.4.13), we find that no record X with positive K^3 and zero P_{24} is found. This proves the theorem.

□

II.5 Appendix to section II.4.

The following table gives the values of Γ_i and Γ_i' for each type of singularity with index less than 26. These are used in the proof of Theorem II.4.6.

No.	Singularity	Γ_1	Γ_2	Γ_3	Γ_4	Γ_1'	Γ_2'	Γ_3'	Γ_4'
1	$\frac{1}{2}(1,1,1)$	1	2	10	112	1	0	0	0
2	$\frac{1}{3}(2,1,1)$	1	3	13	149	0	1	0	0
3	$\frac{1}{4}(3,1,1)$	1	3	15	167	0	1	2	4
4	$\frac{1}{5}(4,1,1)$	1	3	16	178	0	1	3	8
5	$\frac{1}{5}(3,2,1)$	2	5	24	268	1	1	1	0
6	$\frac{1}{6}(5,1,1)$	1	3	16	185	0	1	3	15
7	$\frac{1}{7}(6,1,1)$	1	3	16	190	0	1	3	20
8	$\frac{1}{7}(5,2,1)$	3	7	34	383	2	1	1	3
9	$\frac{1}{7}(4,3,1)$	2	6	28	319	0	2	2	7
10	$\frac{1}{8}(7,1,1)$	1	3	16	194	0	1	3	24
11	$\frac{1}{8}(5,3,1)$	3	8	37	419	1	2	1	2
12	$\frac{1}{9}(8,1,1)$	1	3	16	197	0	1	3	27
13	$\frac{1}{9}(7,2,1)$	4	9	44	497	3	1	1	5
14	$\frac{1}{9}(5,4,1)$	2	6	31	346	0	2	5	13
15	$\frac{1}{10}(9,1,1)$	1	3	16	199	0	1	3	29

No.	Singularity	Γ_1	Γ_2	Γ_3	Γ_4	Γ_1'	Γ_2'	Γ_3'	Γ_4'
16	$\frac{1}{10}(7, 3, 1)$	3	9	41	469	0	3	2	8
17	$\frac{1}{11}(10, 1, 1)$	1	3	16	200	0	1	3	30
18	$\frac{1}{11}(9, 2, 1)$	5	11	54	610	4	1	1	6
19	$\frac{1}{11}(8, 3, 1)$	4	11	50	569	1	3	1	3
20	$\frac{1}{11}(7, 4, 1)$	3	9	43	487	0	3	4	12
21	$\frac{1}{11}(6, 5, 1)$	2	6	32	364	0	2	6	24
22	$\frac{1}{12}(11, 1, 1)$	1	3	16	200	0	1	3	30
23	$\frac{1}{12}(7, 5, 1)$	5	12	58	651	3	2	2	3
24	$\frac{1}{13}(12, 1, 1)$	1	3	16	200	0	1	3	30
25	$\frac{1}{13}(11, 2, 1)$	6	13	64	722	5	1	1	6
26	$\frac{1}{13}(10, 3, 1)$	4	12	54	618	0	4	2	8
27	$\frac{1}{13}(9, 4, 1)$	3	9	46	513	0	3	7	17
28	$\frac{1}{13}(8, 5, 1)$	5	13	61	687	2	3	2	2
29	$\frac{1}{13}(7, 6, 1)$	2	6	32	375	0	2	6	35
30	$\frac{1}{14}(13, 1, 1)$	1	3	16	200	0	1	3	30
31	$\frac{1}{14}(11, 3, 1)$	5	14	63	718	1	4	1	3
32	$\frac{1}{14}(9, 5, 1)$	3	9	47	524	0	3	8	21
33	$\frac{1}{15}(14, 1, 1)$	1	3	16	200	0	1	3	30
34	$\frac{1}{15}(13, 2, 1)$	7	15	74	834	6	1	1	6
35	$\frac{1}{15}(11, 4, 1)$	4	12	58	654	0	4	6	16
36	$\frac{1}{15}(8, 7, 1)$	2	6	32	384	0	2	6	44
37	$\frac{1}{16}(15, 1, 1)$	1	3	16	200	0	1	3	30
38	$\frac{1}{16}(13, 3, 1)$	5	15	67	767	0	5	2	8
39	$\frac{1}{16}(11, 5, 1)$	3	9	48	542	0	3	9	32
40	$\frac{1}{16}(9, 7, 1)$	7	16	78	880	5	2	2	8
41	$\frac{1}{17}(16, 1, 1)$	1	3	16	200	0	1	3	30
42	$\frac{1}{17}(15, 2, 1)$	8	17	84	946	7	1	1	6
43	$\frac{1}{17}(14, 3, 1)$	6	17	76	867	1	5	1	3
44	$\frac{1}{17}(13, 4, 1)$	4	12	61	680	0	4	9	21
45	$\frac{1}{17}(12, 5, 1)$	7	17	82	919	4	3	3	3
46	$\frac{1}{17}(11, 6, 1)$	3	9	48	549	0	3	9	39
47	$\frac{1}{17}(10, 7, 1)$	5	15	69	788	0	5	4	15
48	$\frac{1}{17}(9, 8, 1)$	2	6	32	391	0	2	6	51
49	$\frac{1}{18}(17, 1, 1)$	1	3	16	200	0	1	3	30
50	$\frac{1}{18}(13, 5, 1)$	7	18	85	955	3	4	3	2
51	$\frac{1}{18}(11, 7, 1)$	5	15	71	806	0	5	6	19
52	$\frac{1}{19}(18, 1, 1)$	1	3	16	200	0	1	3	30

No.	Singularity	Γ_1	Γ_2	Γ_3	Γ_4	Γ_1'	Γ_2'	Γ_3'	Γ_4'
53	$\frac{1}{19}(17, 2, 1)$	9	19	94	1058	8	1	1	6
54	$\frac{1}{19}(16, 3, 1)$	6	18	80	916	0	6	2	8
55	$\frac{1}{19}(15, 4, 1)$	5	15	73	821	0	5	8	20
56	$\frac{1}{19}(14, 5, 1)$	4	12	63	702	0	4	11	29
57	$\frac{1}{19}(13, 6, 1)$	3	9	48	560	0	3	9	50
58	$\frac{1}{19}(12, 7, 1)$	8	19	92	1034	5	3	3	6
59	$\frac{1}{19}(11, 8, 1)$	7	19	87	988	2	5	2	5
60	$\frac{1}{19}(10, 9, 1)$	2	6	32	396	0	2	6	56
61	$\frac{1}{20}(19, 1, 1)$	1	3	16	200	0	1	3	30
62	$\frac{1}{20}(17, 3, 1)$	7	20	89	1016	1	6	1	3
63	$\frac{1}{20}(13, 7, 1)$	3	9	48	565	0	3	9	55
64	$\frac{1}{20}(11, 9, 1)$	9	20	98	1107	7	2	2	11
65	$\frac{1}{21}(20, 1, 1)$	1	3	16	200	0	1	3	30
66	$\frac{1}{21}(19, 2, 1)$	10	21	104	1170	9	1	1	6
67	$\frac{1}{21}(17, 4, 1)$	5	15	76	847	0	5	11	25
68	$\frac{1}{21}(16, 5, 1)$	4	12	64	720	0	4	12	40
69	$\frac{1}{21}(13, 8, 1)$	8	21	98	1106	3	5	3	4
70	$\frac{1}{21}(11, 10, 1)$	2	6	32	399	0	2	6	59
71	$\frac{1}{22}(21, 1, 1)$	1	3	16	200	0	1	3	30
72	$\frac{1}{22}(19, 3, 1)$	7	21	93	1065	0	7	2	8
73	$\frac{1}{22}(17, 5, 1)$	9	22	106	1187	5	4	4	3
74	$\frac{1}{22}(15, 7, 1)$	3	9	48	574	0	3	9	64
75	$\frac{1}{22}(13, 9, 1)$	5	15	77	859	0	5	12	30
76	$\frac{1}{23}(22, 1, 1)$	1	3	16	200	0	1	3	30
77	$\frac{1}{23}(21, 2, 1)$	11	23	114	1282	10	1	1	6
78	$\frac{1}{23}(20, 3, 1)$	8	23	102	1165	1	7	1	3
79	$\frac{1}{23}(19, 4, 1)$	6	18	88	988	0	6	10	24
80	$\frac{1}{23}(18, 5, 1)$	9	23	109	1223	4	5	4	2
81	$\frac{1}{23}(17, 6, 1)$	4	12	64	734	0	4	12	54
82	$\frac{1}{23}(16, 7, 1)$	10	23	112	1263	7	3	3	11
83	$\frac{1}{23}(15, 8, 1)$	3	9	48	578	0	3	9	68
84	$\frac{1}{23}(14, 9, 1)$	5	15	78	870	0	5	13	34
85	$\frac{1}{23}(13, 10, 1)$	7	21	95	1087	0	7	4	16
86	$\frac{1}{23}(12, 11, 1)$	2	6	32	400	0	2	6	60
87	$\frac{1}{24}(23, 1, 1)$	1	3	16	200	0	1	3	30
88	$\frac{1}{24}(19, 5, 1)$	5	15	79	880	0	5	14	37
89	$\frac{1}{24}(17, 7, 1)$	7	21	97	1107	0	7	6	22

No.	Singularity	Γ_1	Γ_2	Γ_3	Γ_4	Γ_1'	Γ_2'	Γ_3'	Γ_4'
90	$\frac{1}{24}(13, 11, 1)$	11	24	118	1332	9	2	2	12
91	$\frac{1}{25}(24, 1, 1)$	1	3	16	200	0	1	3	30
92	$\frac{1}{25}(23, 2, 1)$	12	25	124	1394	11	1	1	6
93	$\frac{1}{25}(22, 3, 1)$	8	24	106	1214	0	8	2	8
94	$\frac{1}{25}(21, 4, 1)$	6	18	91	1014	0	6	13	29
95	$\frac{1}{25}(19, 6, 1)$	4	12	64	745	0	4	12	65
96	$\frac{1}{25}(18, 7, 1)$	7	21	99	1125	0	7	8	26
97	$\frac{1}{25}(17, 8, 1)$	3	9	48	585	0	3	9	75
98	$\frac{1}{25}(16, 9, 1)$	11	25	122	1377	8	3	3	13
99	$\frac{1}{25}(14, 11, 1)$	9	25	113	1287	2	7	2	6
100	$\frac{1}{25}(13, 12, 1)$	2	6	32	400	0	2	6	60

II.6 Anti-plurigenera for Q-Fano 3-folds

Assume that X is a $\mathbf{Q}$-Fano 3-fold, (i.e. a projective 3-fold with at worst canonical singularities and $-K_X$ ample). Standard use of vanishing (compare Theorem I.6.2) gives

II.6.1 Theorem: *For all $n \geq 1$*

$$P_{-n} = h^0(\mathcal{O}_X(-nK_X)) = \chi(\mathcal{O}_X(-nK_X))$$

Thus

$$(*) \quad P_{-n} = \frac{(2n+1)n(n+1)}{12}(-K_X)^3 + (1+2n)\chi(\mathcal{O}_X) - l(n+1)$$

where $l(n)$ is defined in Definition II.2.5. Notice that the term involving $l(n)$ is negative.

II.7 The 'Inverse' Problem

Section II.4 gives a formula for calculating the plurigenera of a canonical 3-fold from its record of pluridata. The following solves the "inverse" problem.

II.7.1 Theorem: *Let $P : \mathbf{N} \longrightarrow \mathbf{Z}$ be an arithmetic function which corresponds to a list of plurigenera of some canonical 3-fold X. Then the record of pluridata (K_X^3, p_g, χ, the global index R, and the basket of singularities) can be determined uniquely.*

Proof. This follows from Theorems II.7.3 and II.7.4 below. □

II.7.2 Note. Of course there is nothing special about the role of the plurigenera. The theorem could be rewritten to use either the anti-plurigenera of a $\mathbb{Q}$-Fano 3-fold, or the complete list $\{\chi(\mathcal{O}_X(nK_X))\}$ for a general projective 3-fold with at worst canonical singularities.

The following theorem and its proof show how to calculate the global index.

II.7.3 Theorem: *Let $P : \mathbb{N} \longrightarrow \mathbb{Z}$ be an arithmetic function which corresponds to a list of plurigenera of some canonical 3-fold X. Then K_X^3, p_g, χ, the global index R, and the correction function $l(n)$ can be determined uniquely.*

Proof. Firstly $p_g = P(1)$. By the plurigenera formula, $\delta^3 P(n) = K_X^3 + \delta^3 l(n)$. By Lemma II.3.9, this is of exact period R and so determines R. Now $\delta l(mR) = 0$ for all m and so

$$Q_R = \delta P(R) = \tfrac{1}{2}R^2K_X^3 - 2\chi$$

and

$$Q_{2R} = \delta P(2R) = 2R^2K_X^3 - 2\chi$$

allowing both K_X^3 and χ to be determined. Hence $l(n)$ can also be determined. □

So Theorem II.7.1 has been reduced to decoding the correction function $l(n)$. This is done using the next theorem.

II.7.4 Theorem: *The functions $l(Q, n)$ for each type of terminal quotient 3-fold singularity Q, with index dividing some global index R, are linearly independent.*

II.7.5 Origin of the idea of proof. The proof of this theorem follows a similar proof due to Reid (see [R4, appendix to section 5]). However Reid deals with the linear functions $\overline{bk} - r/2$ and odd characters arise; whereas this section deals with the quadratic functions $\overline{bk}\,(R - \overline{bk})$ (modulo R) and hence with even characters.

Like Reid I shall start with a slightly easier problem (compare [R4, Proposition 5.9]).

II.7.6 Lemma: *For fixed index r, the functions $l(Q, n)$ for each type of terminal quotient singularity $Q = \dfrac{1}{r}(1, -1, a)$, with $\mathrm{hcf}(r, a) = 1$, are linearly independent.*

II.7.7 Well-known Results. Let $\phi(r)$ be Euler's function (i.e. the order of $\mathbb{Z}_r^*$). Then we have the following from [H&W, section V.5.5]:

(1) $\phi(1) = 1 = \phi(2)$,

(2) $2 | \phi(r)$ for all $r \geq 3$ (see [H&W, Theorem 62]),

(3) $R = \sum_{r|R} \phi(r)$ (see [H&W, Theorem 63]).

The following is deduced from the above 3 facts.

$$\left\lfloor \frac{R}{2} \right\rfloor = \sum_{r|R,\, r\geq 2} \left\lfloor \frac{\phi(r)+1}{2} \right\rfloor .$$

Notice that for a fixed index $r > 2$ there are $\phi(r) / 2$ types of singularity since the types $\frac{1}{r}(1,-1,a)$ and $\frac{1}{r}(1,-1,r-a)$ are equivalent.

II.7.8 Proof of Lemma II.7.6. If $r = 1$ or $r = 2$ the result is trivial. Without loss of generality assume that $r \geq 3$. There are $\phi(r)/2$ such types of singularity of index r and

$$l(Q,n) = l(\frac{1}{r}(1,-1,b),n) = \sum_{k=0}^{n-1} \frac{\overline{nb}\,(r-\overline{nb})}{2r}$$

with b coprime to r. Clearly these correction functions are linearly independent if and only if the functions

$$2r\delta l\,(Q,n) = \overline{nb}(r-\overline{nb})$$

are. In fact we need only consider the $\phi(r)$ vectors in $\mathbf{Z}^{r-1}$:

$$T_b = (\overline{kb}(r-\overline{kb}))_{k=1,\ldots,r-1}$$

for b coprime to r. Let $\mathcal{V}$ be the $\mathbf{C}$-vector space spanned by these vectors. Note that

$$(T_a)_k = (T_{r-a})_k = (T_a)_{r-k} = (T_{r-a})_{r-k}$$

and so $\dim_{\mathbf{C}} \mathcal{V} \leq \phi(r) / 2$.

Let G be the the group of Dirichlet characters:

$$\chi : \mathbf{Z}_r^* \longrightarrow \mathbf{C}^*$$

and let G_{even} be the even characters (i.e. those characters χ such that $\chi(-1) = 1$). By [Wash, Lemma 3.1], $|G| = \phi(r)$ and $|G_{even}| = \phi(r) / 2$.

For each character χ define

$$W_\chi = \sum_{a \in \mathbf{Z}_r^*} \chi(a) T_a$$

Note that $W_\chi = 0$ for odd χ. In comparison with [R4, appendix to section 5] Reid finds that in his case, $W_\chi = 0$ for even χ.

Let 1 be the trivial character.

$$\begin{aligned}(W_1)_1 &= \sum_{a \in \mathbf{Z}_r^*} (T_a)_1 \\ &= \sum_{a \in \mathbf{Z}_r^*} a(r-a) > 0\end{aligned}$$

So $W_1 \neq 0$.

Consider a non-trivial character χ with conductor $f \neq 1$. The following commutes

$$\begin{tikzcd} \mathbf{Z}_r^* \arrow[rr, "\chi"] \arrow[dr, "\sigma"'] & & \mathbf{C}^* \\ & \mathbf{Z}_f^* \arrow[ur, "\chi'"'] & \end{tikzcd}$$

where σ is the projection mod f. So $1 = \chi(-1) = \chi'\sigma(-1) = \chi'(-1)$. Hence $\chi' : \mathbf{Z}_f^* \longrightarrow \mathbf{C}^*$ is an even character. Let $q = r/f$.

$$\begin{aligned}(W_\chi)_q &= \sum_{a \in \mathbf{Z}_r^*} \chi(a)(T_a)_q \\ &= \sum_{a \in \mathbf{Z}_r^*} \chi(a)\, \overline{aq}\,(r - \overline{aq})\end{aligned}$$

But $(T_a)_q = (T_{a+f})_q$, and so depends only on a mod f. Thus

$$\begin{aligned}(W_\chi)_q &= \frac{\phi(r)}{\phi(f)} \sum_{a' \in \mathbf{Z}_f^*} \chi'(a')\, a'q(r - a'q) \\ &= \frac{\phi(r)}{\phi(f)} q^2 \sum_{a' \in \mathbf{Z}_f^*} \chi'(a')\, a'(f - a') \\ &= \frac{\phi(r)}{\phi(f)} q^2 \left[\frac{f^2}{6} \sum_{a \in \mathbf{Z}_f^*} \chi'(a) - \sum_{a \in \mathbf{Z}_f^*} \chi'(a)(a^2 - fa + f^2/6) \right]\end{aligned}$$

Let $B_{n,\chi}$ be the generalized Bernoulli numbers as defined in [Wash, p. 30]. Then

$$B_{2,\chi'} = \sum_{a \in \mathbf{Z}_f^*} \chi'(a) \frac{(a^2 - af + f^2/6)}{f}.$$

Also $\sum \chi'(a) f^2/6 = 0$ since $f^2/6$ is a constant. So

$$(W_\chi)_q = -\frac{\phi(r)}{\phi(f)} q^2 f B_{2,\chi'} = -\frac{\phi(r)}{\phi(f)} qr B_{2,\chi'}.$$

See also [Wash, Exercise 4.2 (a)]. By [Wash, Theorem 4.2 and p. 30],

$$B_{2,\chi'} = -2L(-1, \chi') \neq 0$$

for even χ. Thus $W_\chi \neq 0$ for non-trivial χ.

Let $\mathbf{Z}_r^*$ act on $\mathbf{C}^{\phi(r)}$ by permuting the coordinates. Let b be a generator. Then $(bx)_k = (x)_{bk}$. So

$$bT_a = T_{ab}$$

and

$$bW_\chi = \chi(b)^{-1} W_\chi$$

for all even characters χ. As $\chi(b)$ are distinct (or else the characters would not be distinct) and the W_χ are non-zero, then they are eigenvectors of b. Therefore the vectors W_χ are linearly independent. Hence $\dim_{\mathbf{C}} \mathcal{V} \geq \phi(r)/2$ and so $\dim_{\mathbf{C}} \mathcal{V} = \phi(r)/2$. Thus the original vectors T_a are linearly independent.

$\square$

II.7.9 The proof of Theorem II.7.4.

The above proof does not generalise to Theorem II.7.4 since there are not enough characters. The proof of II.7.4 will done in a number of stages, and involves 2 changes of 'basis'. The main steps in the proof are Theorems II.7.14 and II.7.15, and sections II.7.16 and II.7.23.

There are 3 sets of bases $\{T_a\}$, $\{W_\chi(a)\}$ and $\{V_\chi(a)\}$ used, defined in Definitions II.7.10, II.7.11 and II.7.18 respectively. Lemmas II.7.12, II.7.20, II.7.21 and II.7.22 are technical results on the vanishing and non-vanishing of certain coordinates of $W_\chi(a)$ and $V_\chi(a)$.

As before in the proof of Lemma II.7.6, we consider the following vector space.

II.7.10 Definition: *Let* $T_a = (\overline{ak}(R-\overline{ak}))_{k=1,\dots,R-1}$. *Let* $\mathcal{V}$ *be the subspace of* $\mathbf{C}^{R-1}$ *spanned by these vectors.*

As in the previous proof, $\dim_{\mathbf{C}} \mathcal{V} \leq \lfloor R/2 \rfloor$. Clearly the vectors $\{T_a : a=1, \dots, \lfloor R/2 \rfloor\}$ are linearly independent if and only if Theorem II.7.4 is true.

II.7.11 Definition: *Let* $\chi : \mathbf{Z}_f^* \longrightarrow \mathbf{C}^*$ *be an even Dirichlet character with conductor* f *and* $q = R/f$. *Define*

$$W_\chi(a) = \sum_{b \in \mathbf{Z}_f^*} \chi(b) T_{ab} \in \mathcal{V}$$

II.7.12 Lemma:

(i) *The* cth *coordinate* $(W_\chi(a))_c$ *depends only on* $ac \bmod R$

(ii) *If* $ac = q$ *then* $(W_\chi(a))_c = -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi} \neq 0$.

(iii) *If* $\beta \in \mathbf{Z}_f^*$ *then* $W_\chi(\beta a) = \chi(\beta)^{-1} W_\chi(a)$

(iv) *If* $a | R$ *and* $\mathrm{hcf}(ac, R) \nmid q$ *then* $(W_\chi(a))_c = 0$.

Proof.

(i) $(W_\chi(a))_c = \sum_{b \in \mathbf{Z}_f^*} \chi(b) \overline{acb}\,(R - \overline{acb})$, which depends only on $ac \bmod R$.

(ii)

$$\begin{aligned}(W_\chi(a))_c &= (W_\chi(1))_{ac} \\ &= (W_\chi(1))_q \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi} \neq 0\end{aligned}$$

(Compare with the proof of Lemma II.7.6.)

(iii)

$$\begin{aligned} W_\chi(\beta a) &= \sum_{b \in \mathbb{Z}_f^*} \chi(b) T_{a\beta b} \\ &= \chi(\beta)^{-1} \sum_b \chi(\beta b) T_{a\beta b} \\ &= \chi(\beta)^{-1} W_\chi(a). \end{aligned}$$

(iv) Let $q' = \mathrm{hcf}(ac, R)$ and $f' = R/q'$. Then there exists β coprime to R such that

$$\begin{aligned} (W_\chi(a))_c &= \chi(\beta) \sum_b \chi(b)(T_1)_{q'} \\ &= \chi(\beta) \sum_b \chi(b) Q(bq'). \end{aligned}$$

The function $Q(bq')$ depends only on b mod f' and so

$$Q(kbq') = Q(bq')$$

for all $k \in K = \mathrm{Ker}(\mathbb{Z}_f^* \longrightarrow \mathbb{Z}_{f'}^*)$.

Assume that χ is trivial on K. Then K is contained in $\mathrm{Ker}(\mathbb{Z}_f^* \longrightarrow \mathbb{Z}_{f'}^*)$ (since f is the conductor). So $f | f' | R$ (i.e. $1 | q' | q$). But $q' = \mathrm{hcf}(ac, R) \nmid q$, a contradiction. Thus χ is not trivial on K. So

$$\begin{aligned} (W_\chi(a))_c &= \sum_b \chi(b) Q(bq') \\ &= \sum_{k \in K} \sum_{b' \in \mathbb{Z}_{f'}^*} \chi(kb') Q(b'q') \\ &= \sum_{b' \in \mathbb{Z}_{f'}^*} Q(b'q') \sum_{k \in K} \chi(kb') = 0. \end{aligned}$$

□

II.7.13 Note. For all a in $\mathbb{Z}_R$ there exists a β coprime to R such that

$$W_\chi(a) = \chi(\beta)^{-1} W_\chi(\mathrm{hcf}(a, R)).$$

II.7.14 Theorem: *The subspace of $\mathbb{C}^{R-1}$ generated by the set*

$$\left\{ W_\chi(a) : \chi \text{ even characters of } \mathbb{Z}_R^* \text{ and } a \in \mathbb{Z}_R \text{ such that } a \,\Big|\, \frac{R}{f_\chi} \right\}$$

lies in $\mathcal{V}$ and splits into $\phi(R)$ distinct eigenspaces, one for each χ. There are $\left\lfloor \frac{R}{2} \right\rfloor$ vectors in the above set.

Proof. By Lemma II.7.12 (*ii*), each vector in the above set is a non-zero sum of the vectors $\{T_a\}$. Each $W_\chi(a)$ is an eigenvector with eigenvalue χ^{-1} under the action of the group $\mathbf{Z}_R^*$.

□

Fix the character χ once and for all and consider the χ-eigenspace.

II.7.15 Theorem: *For a fixed character χ the vectors $\{W_\chi(a):a|q\}$ are linearly independent, where $q = R/f_\chi$.*

This will be proved after some preliminary work.

II.7.16 Proof of Theorem II.7.4. Theorems II.7.14 and II.7.15 imply that $\{W_\chi(a)\}$ are linearly independent and hence so are vectors $\{T_a\}$. This proves Theorem II.7.4, subject to proving Theorem II.7.15.

□

To prove Theorem II.7.15 the following definition and another change of basis is required.

II.7.17 Definition: *Let $\mathcal{P}$ be the set of primes which divide q but not f. For each $p \in \mathcal{P}$ define β_p by*

$$\beta_p = p \mod R/p^\alpha$$
$$\beta_p = 1 \mod p^\alpha$$

where p^α is the highest power of p dividing R. These 2 equations have a unique common solution modulo R.

Extend this definition to the set $\mathcal{D}$ of products of distinct primes in $\mathcal{P}$

$$\beta_d = \Pi \beta_{p_i}$$

where $d = \Pi p_i \in \mathcal{D}$.

II.7.18 Note.

(i) $\beta_p \in \mathbf{Z}_R^*$ since $\mathrm{hcf}(\beta_p, p^\alpha)=1$ and $\mathrm{hcf}(\beta_p, R)|p$.

(ii) $\beta_p x \equiv px \mod R$ whenever $p^{\alpha+1}|x$.

We now make the second change of basis.

II.7.19 Definition: *For all $a|q$, define*

$$V_\chi(a) = \sum_{d \in \mathcal{D}:\, d|a} \mu(d)\chi^{-1}(\beta_d)W_\chi(a/d)$$
$$= \sum_{d \in \mathcal{D}:\, d|a} \mu(d)W_\chi(\beta_d a/d)$$

where $\mu(d)$ is the Möbius function (i.e. $\mu(d) = (-1)^m$, where d is a product of m distinct primes, and $\mu(d) = 0$ if $p^2|d$ for some prime p).

II.7.20 Lemma: *Let $a, c \mid q$ but $ac \nmid q$. Then $(V_\chi(a))_c = 0$.*

Proof. As $ac \nmid q$ there is a prime p such that $p^\gamma \mid ac$ but $p^\gamma \nmid q$. There are 2 cases:

(i) $p \notin \mathcal{P}$ (i.e. $p \mid f$).

Then $p^\gamma \mid \mathrm{hcf}(ac, R)$ and so $\mathrm{hcf}(ac, R) \nmid q$. By Lemma II.7.12 (*iv*), $(W_\chi(a))_c = 0$. Similarly $(W_\chi(\beta_d a/d))_c = \chi^{-1}(\beta_d)(W_\chi(a/d))_c$ and $p^\gamma \mid ac/d$ (since $p \notin \mathcal{P}$).
Thus $(W_\chi(\beta_d a/d))_c = 0$ and so $(V_\chi(a))_c = 0$.

(ii) $p \in \mathcal{P}$ (i.e. $p \nmid f$).

By the careful grouping of terms,

$$(V_\chi(a))_c = \sum_{d \mid a,\, p \nmid d} \mu(d)(W_\chi(\beta_d a/d))_c + \mu(pd)(W_\chi(\beta_d \beta_p a/pd))_c$$

$$= \sum_{d \mid a,\, p \nmid d} \mu(d)\Big[(W_\chi(\beta_d a/d))_c - (W_\chi(\beta_d \beta_p a/pd))_c\Big]$$

Notice that $p^\alpha \mid ac$ but $p^{\alpha+1} \nmid ac$. By Note II.7.18 (*ii*),

$$\beta_p \frac{ac}{pd} \equiv p\frac{ac}{pd} \equiv \frac{ac}{d} \mod R$$

So each pair of terms cancels out to give $(V_\chi(a))_c = 0$.

$\square$

II.7.21 Lemma: *Let $ac = q$ and $d \mid a$ for some $d \in \mathcal{D}$. Then*

$$(W_\chi(\beta_d a/d))_c = -\frac{\phi(R)}{\phi(f)} qRB_{2,\chi'} \prod_{p \mid d} \frac{\chi(p)^{-1} - p}{p(p-1)}.$$

Proof. Define $Q(x) = \bar{x}(R - \bar{x})$. By definition

$$(W_\chi(\beta_d a/d))_c = (W_\chi(1))_{\beta_d q/d}$$

$$= \sum_{b \in \mathbb{Z}_f} \chi(b) Q(b\beta_d q/d)$$

$$= \frac{\phi(R)}{\phi(f)} \sum_{b' \in \mathbb{Z}_f} \chi(b') \underset{b \in \mathbb{Z}_q^*}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d)$$

Let $t(d') = \underset{b \in d'\mathbb{Z}_q^*}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d)$, where $d' \mid d$. Since

$$\mathbb{Z}_q = \mathbb{Z}_q^* \cup \bigcup_{\substack{d' \mid d \\ d' > 1}} d'\mathbb{Z}_q$$

then

$$\underset{b \in \mathbb{Z}_q^*}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d) = \underset{b \in \mathbb{Z}_q}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d) - \sum_{\text{prime } p \mid d} \underset{b \in p\mathbb{Z}_q}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d)$$
$$+ \sum_{p_1 p_2 \mid d} \underset{b \in p_1 p_2 \mathbb{Z}_q}{\longmapsto\!\!\sum} b'\, Q(b\beta_d q/d) - \ldots$$

$$= \sum_{d'|d} \mu(d') \sum_{b\in d'\mathbb{Z}_{df}} b' Q(b\beta_d q/d)$$

$$= \sum_{d'|d} \mu(d') s(d').$$

Consider the sum $s(d')$. As d and f are coprime there is a unique integer $i_0 < d$ such that $dx = b' + i_0 f$ for some x. Since the sum $s(d')$ involves only $b \in d'\mathbb{Z}_{df}$ and $Q(\beta_d bq/d)$ depends only on b mod f then

$$s(d') = \sum_{j=0}^{d''-1} Q((b' + i_0 f + jd'f)\beta_d q/d)$$

where $d'' = d/d'$. However $(b' + i_0 f)\beta_d q/d \equiv b'q$ mod R (by definition of β_d). So

$$\begin{aligned} s(d') &= \sum_{j=0}^{d''-1} Q(b'q + jd'\beta_d f q/d) \\ &= \sum_{j=0}^{d''-1} Q(b'q + jd'R/d) \\ &= \sum_{j=0}^{d''-1} Q(b'q + jR/d'') \end{aligned}$$

The numbers $\{\overline{b'q + jR/d''} : j=0, \ldots, d''-1\}$ take their smallest value of $(\overline{\overline{b'd''}} \cdot q/d'')$ where $\overline{\overline{h}}$ denotes least positive residue mod f. Thus the range of summation can be rewritten;

$$s(d') = \sum_{i=0}^{d''-1} Q(\overline{\overline{b'd''}} \cdot q/d'' + iR/d'')$$

Notice that $Q(a + b) = Q(a) + Q(b) - 2ab$ for $a + b < R$. So

$$\begin{aligned} s(d'') &= \sum_{i=0}^{d''-1} \left[Q(\overline{\overline{b'd''}} \cdot q/d'') + Q(iR/d'') - 2\,\overline{\overline{b'd''}} \cdot iRq/d''^2 \right] \\ &= d''Q(\overline{\overline{b'd''}} \cdot q/d'') + \sum_{i=0}^{d''-1} Q(iR/d'') - 2\frac{qR}{d''^2}\binom{d''}{2} \cdot \overline{\overline{b'd''}}. \end{aligned}$$

The calculation of $\sum_{b'\in\mathbb{Z}_f^*} \chi(b') \sum_{d'|d} \mu(d') s(d')$ consists of 3 parts:

(i)

$$\sum_{d'|d} \sum_{b'\in\mathbb{Z}_f^*} \chi(b')\mu(d') \left[\sum_{j=0}^{d''-1} Q(iR/d'') \right] = 0,$$

since the summand is independent of b'.

(ii)

$$\sum_{d'|d} \sum_{b'\in\mathbb{Z}_f^*} \chi(b')\mu(d')\binom{d''}{2}\frac{2qR}{d''^2} \cdot \overline{\overline{b'd''}} = \sum_{d'|d} \sum_{b'\in\mathbb{Z}_f^*} \chi(b')\mu(d')\binom{d''}{2}\frac{2qR}{d''^2} \cdot \overline{\overline{b'}} = 0,$$

since d'' and f are coprime.

(iii)

$$\sum_{d'|d}\sum_{b' \in \mathbb{Z}_f} \chi(b')\mu(d')d''Q(\overline{b'd''}\cdot q/d'') = \sum_{d'|d}\sum_{b' \in \mathbb{Z}_f} \chi(b')\mu(d')d''\frac{q}{d''}\cdot\overline{b'd''}\cdot\left(R - \frac{q}{d''}\cdot\overline{b'd''}\right)$$

$$= \sum_{d'|d}\frac{q^2}{d''}\mu(d')\sum_{b' \in \mathbb{Z}_f}\chi(b')\cdot\overline{b'd''}\cdot(fd'' - \overline{b'd''})$$

$$= \sum_{d'|d}\frac{q^2}{d''}\mu(d')\chi(d'')^{-1}\sum_{b' \in \mathbb{Z}_f}\chi(b')b'(fd'' - b')$$

$$= \sum_{d'|d}\frac{q^2}{d''}\mu(d')\chi(d'')^{-1}\sum_{b' \in \mathbb{Z}_f}\chi(b')\left[b'(f-b') + b'f(d''-1)\right]$$

$$= -\sum_{d'|d}\frac{q^2}{d''}\mu(d')\chi(d'')^{-1}fB_{2,\chi'}.$$

So $\sum_{b' \in \mathbb{Z}_f}\chi(b')\sum_{d'|d}\mu(d')t(d')$ is the sum of these 3 parts. Therefore

$$(W_\chi(\beta_d q/d))_c = -\frac{\phi(R)}{\phi(q)}RqB_{2,\chi'}\sum_{d'|d}\frac{\mu(d')}{d''}\chi(d'')^{-1}$$

$$= -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\frac{\chi(d)^{-1}}{d\phi(d)}\sum_{d'|d}d'\mu(d')\chi(d'')^{-1}$$

$$= -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\frac{\chi(d)^{-1}}{d\phi(d)}\prod_{p|d}\left[1 - p\chi(p)\right]$$

$$= -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\prod_{p|d}\left[\frac{\chi(p)^{-1} - p}{p(p-1)}\right]$$

$$= -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\prod_{p|d}\left[1 - \frac{p^2 - \chi(p)^{-1}}{p(p-1)}\right]$$

$\square$

II.7.22 Lemma: *Let $ac = q$. Then*

$$(V_\chi(a))_c = -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\prod_{p|a}\frac{p^2 - \chi(p)^{-1}}{p(p-1)} \neq 0.$$

Proof.

$$(V_\chi(a))_c = \sum_{d|a}(W_\chi(\beta_d a/d))_c$$

$$= -\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\sum_{d|a}\left[\mu(d)\prod_{p|d}\left[1 - \frac{p^2 - \chi(p)^{-1}}{p(p-1)}\right]\right]$$

But

$$\prod_{p|a}\frac{p^2-\chi(p)^{-1}}{p(p-1)}=\prod_{p|a}\left[1-\frac{\chi(p)^{-1}-p}{p(p-1)}\right]$$
$$=\sum_{d|a}\mu(d)\prod_{p|d}\frac{\chi(p)^{-1}-p}{p(p-1)}$$

Therefore

$$(V_\chi(a))_c=-\frac{\phi(R)}{\phi(f)}RqB_{2,\chi'}\prod_{p|a}\frac{p^2-\chi(p)^{-1}}{p(p-1)}.$$

Clearly $\chi(p)^{-1}\neq p^2$ and so $(V_\chi(a))_c\neq 0$.

□

II.7.23 Proof of Theorem II.7.15. Clearly the vectors $\{V_\chi(a):a|q\}$ lie in the subspace spanned by $\{W_\chi(a):a|q\}$. Let $\{a_i:i=1,\ldots,n\}$ be the set of $a|q$, ordered such that $a_i\nmid a_j$ for $i>j$. Let $c_i=q/a_i$. Let M be the matrix with entries

$$M_{i,j}=(V_\chi(a_i))_{c_j}$$

Suppose $i>j$. Then $a_i|q$, $c_j|q$ and $a_ic_j\nmid q$. By Lemma II.7.20, $M_{i,j}=0$. So M is an upper triangular matrix. By Lemma II.7.22, the diagonal enties $M_{i,i}$ are non-zero.

Thus M has maximal rank and the vectors $\{V_\chi(a):a|q\}$ are linearly independent. So the vectors $\{W_\chi(a):a|q\}$ also are independent. This completes the proof of Theorem II.7.15.

□

III

Weighted hypersurfaces and complete intersections.

III.1 Preamble

In this chapter we aim to prove necessary and sufficient conditions for a weighted hypersurface X_d in $\mathbb{P}(a_0, ..., a_n)$ to be quasismooth, well-formed and to have only isolated canonical quotient singularities, and to produce lists of such hypersurfaces. Also there will be necessary and sufficient conditions and lists for the codimension 2 case.

Section 2 recaps the main definitions and theorems about weighted complete intersections. Section 3 contains necessary and sufficient conditions for quasismoothness in the hypersurface and codimension 2 cases and some technical results. An example of how to calculate the singularities of a weighted hypersurface is included at the end of the section (see III.3.19).

Sections 4 and 5 treat the cases of dimension 1 and 2 respectively; and give corresponding lists.

Section 6 deals with the 3-fold case (both hypersurfaces and codimension 2) and sections 7 and 8 deal with the particular cases of canonical 3-folds and $\mathbb{Q}$-Fano 3-folds respectively. Section 6 gives a worked example of the determination of the singularities on a codimension 2 weighted complete intersection.

Section 9 gives an alternative method for producing canonically and anticanonically embedded 3-fold complete intersections using the Poincaré series of a ring.

Section 10 contains a selection of computer programs used in the search for quasismooth weighted complete intersections with at worst isolated canonical singularities.

III.2 Definitions and theorems on weighted projective spaces.

We start by reviewing some definitions and notations about weighted complete intersections from Dolgachev [WPS].

III.2.1 Definition: *Let $a_0, \dots, a_n$ be positive integers and define $S = S(a_0, \dots, a_n)$ to be the graded polynomial ring $\mathbf{k}[x_0, \dots, x_n]$, graded by $\deg x_i = a_i$. The weighted projective space $\mathbf{P}(a_0, \dots, a_n)$ is defined by*

$$\mathbf{P}(a_0, \dots, a_n) = \operatorname{Proj} S$$

III.2.2 Note. Let $x_0, \dots, x_n$ be affine coordinates on $\mathbf{A}^{n+1}$. $\mathbf{P}(a_0, \dots, a_n)$ is the quotient $(\mathbf{k}^{n+1} - \underline{0}) / \mathbf{k}^*$ where the group $\mathbf{k}^*$ acts via:

$$\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n).$$

$x_0, \dots, x_n$ under this group action are the homogeneous coordinates on $\mathbf{P}(a_0, \dots, a_n)$. Clearly $\mathbf{P}(a_0, \dots, a_n)$ is a rational n-dimensional projective variety.

III.2.3 Affine coordinates pieces.

Let $\{x_0, \dots, x_n\}$ be the homogeneous coordinates on $\mathbf{P}(a_0, \dots, a_n)$. The affine piece $x_i \neq 0$ is isomorphic to $\mathbf{A}^n / \mathbf{Z}_{a_i}$. Let ε be a primitive a_ith root of unity. The group acts via:

$$z_j \longmapsto \varepsilon^{a_j} z_j$$

for all $j \neq i$, on the coordinates $\{z_0, \dots, \hat{z_i}, \dots, z_n\}$ of $\mathbf{A}^n$; here z_i is thought as $x_j / \sqrt[a_j]{x_i}$. Compare the case of $\mathbf{P}^n$ where the affine coordinates on $x_i \neq 0$ are $z_i = x_j / x_i$.

III.2.4 Examples.

(i) $\mathbf{P}^n = \mathbf{P}(1, \dots, 1)$.

(ii) Consider $\mathbf{P}(1, 1, 2)$ with homogeneous coordinates u, v and w. The affine piece $w = 1$ is $\mathbf{A}^2 / \mathbf{Z}_2$ with group action

$$u \longmapsto -u$$
$$v \longmapsto -v$$

The coordinate ring R is given by:

$$\begin{aligned} R &= \mathbf{k}[u, v]^{\mathbf{Z}_2} \\ &= \mathbf{k}[u^2, v^2, uv] \\ &= \mathbf{k}[x, y, z] / (xy - z^2) \end{aligned}$$

So $\mathbf{P}(1, 1, 2)$ is the projective completion of the ordinary quadratic cone $xy = z^2$ in $\mathbf{A}^3$.

III.2.5 Lemma: *For all positive integers q we have*

$$\operatorname{Proj} S(a_0, \ldots, a_n) \cong \operatorname{Proj} S(qa_0, \ldots, qa_n)$$

Proof. This follows from the fact that the 2 graded rings are isomorphic. $\square$

From [EGA, Proposition 2.4.7] (also see [Hart, Exercise II.5.13]) we have:

III.2.6 Lemma: *Let S be a graded ring and define the truncation $S^{(q)} = \bigoplus_{m \geq 0} S_{qm}$ to be the graded subring, with mth graded part S_{qm}. There exists a canonical isomorphism* $\operatorname{Proj} S^{(q)} \cong \operatorname{Proj} S$.

This is called the q-tuple Veronese embedding, and is used in the proof of the following:

III.2.7 Lemma: *Let $a_0, \ldots, a_n$ be positive integers with no common factor. If* $q = \operatorname{hcf}(a_1, \ldots, a_n)$ *then*

$$\operatorname{Proj} S(a_0, \ldots, a_n) \cong \operatorname{Proj} S(a_0, a_1/q, \ldots, a_n/q)$$

Proof. Define $S' = \bigoplus_{m \geq 0} S_{qm}$ with the same grading as S. So $S' \cong S^{(q)}$. By the previous lemma, $\operatorname{Proj} S' \cong \operatorname{Proj} S$.

Suppose $x_0^{p_0} \ldots x_n^{p_n}$ is a monomial of degree mq for any m. Hence $p_0 a_0 + \ldots + p_n a_n = qm$, and so $q | p_0 a_0$. As the $\{a_i\}$ have no common factor, $q | p_0$. Hence x_0 only appears in S' as x_0^q. Thus $S' = \mathbf{k}[x_0^q, x_1, \ldots, x_n]$, which is isomorphic to $S(qa_0, a_1, \ldots, a_n)$. Therefore

$$\operatorname{Proj} S(a_0, \ldots, a_n) \cong \operatorname{Proj} S' \cong \operatorname{Proj} S(a_0, a_1/q, \ldots, a_n/q)$$

$\square$

This leads to the following corollary from [WPS, 1.3.1] (see also [De, Proposition 1.3]):

III.2.8 Corollary: $\mathbf{P}(a_0, \ldots, a_n) \cong \mathbf{P}(b_0, \ldots, b_n)$ *for some $\{b_i\}$ such that for each i*

$$\operatorname{hcf}(b_0, \ldots, \hat{b}_i, \ldots, b_n) = 1.$$

Proof. By Lemma II.2.5 we can cancel any common factor of the $\{a_i\}$. By renumbering as necessary and repeated applications of Lemma II.2.7 we can reduce $\mathbf{P}(a_0, \ldots, a_n)$ to the case $\mathbf{P}(b_0, \ldots, b_n)$. A maximum of $n+1$ applications of Lemma II.2.7 are required. $\square$

III.2.9 Examples.

(i) $\mathbf{P}(a, b) \cong \mathbf{P}^1$ for all a and b.

(ii) $\mathbf{P}(2, 3, 3) \cong \mathbf{P}(2, 1, 1)$.

(iii) Let $f = x^5 + y^3 + z^2 \in \mathbf{k}[x, y, z]$ with grading 6, 10 and 15 respectively. Define $X : (f = 0) \subset \mathbf{P} = \mathbf{P}(6, 10, 15)$. By the previous lemma $\mathbf{P} \cong \mathbf{P}^2$.

$$\mathbf{P}(6, 10, 15) \cong \mathbf{P}(6, 2, 3) \cong \mathbf{P}(3, 1, 3) \cong \mathbf{P}(1, 1, 1)$$

The monomials transform as:

$$(x^5, y^3, z^2) \longrightarrow (x, y^3, z^2) \longrightarrow (x, y^3, z) \longrightarrow (x, y, z)$$

Thus $X \subset \mathbf{P} \cong (x + y + z = 0) \subset \mathbf{P}^2 = \mathbf{P}^1 \subset \mathbf{P}^2$. Of course the coordinate rings of the affine cones (see III.2.14) over $X \subset \mathbf{P}$ and $\mathbf{P}^1 \subset \mathbf{P}^2$ are not isomorphic.

In view of Corollary III.2.8 we make the following:

III.2.10 Definition: *The expression* $\mathbf{P}(a_0, \ldots, a_n)$ *is well-formed if for each* i

$$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1.$$

III.2.11 The quotient map. Let $T = \mathbf{k}[y_0, \ldots, y_n]$, where the $\{y_i\}$ all have weight 1, and so$\mathbf{P}^n \cong \mathrm{Proj}\, T$. Consider the inclusion map $S \longrightarrow T$ given by:

$$x_i \longmapsto y_i^{a_i}$$

for all i. This induces a quotient map $\sigma : \mathbf{P}^n \longrightarrow \mathbf{P}$. In terms of the coordinates (Y_i) on $\mathbf{P}^n$

$$[Y_0, \ldots, Y_n] \longmapsto [Y_0^{a_0}, \ldots, Y_n^{a_n}]$$

The map $\mathbf{P}^n \longrightarrow \mathbf{P}$ is a ramified Galois covering with Galois group $\bigoplus_i \mathbf{Z}_{a_i}$.

III.2.12 Notation. Write $P_i \in \mathbf{P}$ for the point $[0, \ldots, 0, 1, 0, \ldots, 0]$, where the 1 is in the ith position. We will call P_i a vertex, the 1-dimensional toric stratum P_iP_j an edge, etc.. The fundamental simplex (i.e. the union of all the coordinate hyperplanes $P_0 \ldots \hat{P}_i \ldots P_n$) will be denoted by Δ.

III.2.13 Note. Define $h_{i,j,\ldots} = \mathrm{hcf}(a_i, a_j, \ldots)$. The vertex P_i is a singularity of type $\frac{1}{a_i}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$. This singularity is not necessarily isolated. Each generic point P of the edge P_iP_j has an analytic neighbourhood $P \in U$ which is analytically isomorphic to $(0, Q) \in \mathbf{A}^1 \times Y$, where $Q \in Y$ is a singularity of type $\frac{1}{h_{i,j}}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n)$. Similarly for higher dimensional toric strata. The singularities only occur on the fundamental simplex Δ.

III.2.14 Definition: *Let X be a closed subvariety of a weighted projective space* $\mathbf{P}$, *and let* $p : \mathbf{A}^{n+1} - 0 \longrightarrow \mathbf{P}$ *be the canonical projection. The punctured affine cone C_X over X is given by* $C_X^* = p^{-1}(X)$, *and the affine cone over X is the completion of C_X^* in* $\mathbf{A}^{n+1}$.

Notice that $\mathbf{k}^*$ acts on C_X^* to give $X = C_X^*/\mathbf{k}^*$.

III.2.15 Lemma: *C_X has no isolated singularities.*

Proof. If $P \in C_X$ is singular then every point on the same fibre of the $\mathbf{k}^*$-action will be singular.

$\square$

III.2.16 Definition: *X in $\mathbb{P}(a_0, ..., a_n)$ is quasismooth of dimension m if its affine cone C_X is smooth of dimension $m+1$ outside its vertex $\underline{0}$.*

When $X \subset \mathbb{P}$ is quasismooth the singularities of X are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities.

III.2.17 Definition: *Let I be a homogeneous ideal of the graded ring S and define X_I to be:*

$$X_I = \operatorname{Proj} S/I \subset \mathbb{P}$$

Suppose furthermore that I is generated by a regular sequence $\{f_i\}$ of homogeneous elements of S. $X_I \subset \mathbb{P}$ is called a weighted complete intersection of multidegree $\{d_i = \deg f_i\}$. In this case, we denote by $X_{d_1, ..., d_c}$ in $\mathbb{P} = \mathbb{P}(a_0, ..., a_n)$ a sufficiently general element of the family of all weighted complete intersections of multidegree $\{d_i\}$.

$X_{d_1, ..., d_c}$ in $\mathbb{P}(a_0, ..., a_n)$ is of dimension $n-c$. In general we will write $C_{d_1, ..., d_c}$ in $\mathbb{P}(a_0, ..., a_{c+1})$ for a dimension 1 complete intersection and $S_{d_1, ..., d_c}$ in $\mathbb{P}(a_0, ..., a_{c+2})$ for a surface.

III.2.18 Example. X_{46} in $\mathbb{P}(4, 5, 6, 7, 23)$ is a general element in the family of all degree 46 hypersurfaces in $\mathbb{P}(4, 5, 6, 7, 23)$.

III.2.19 The coefficient convention. When a general polynomial of a given weighted homogeneous degree occurs in a calculation then it will usually be written without the non-zero coefficients. For example the defining polynomial for X_2 in $\mathbb{P}(1, 1, 1)$ is:

$$f = c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2$$

and will be written simply as:

$$f = x^2 + xy + xz + y^2 + yz + z^2.$$

III.2.20 Definition: *A subvariety $X \subset \mathbb{P}$ of codimension c is well-formed if the expression for $\mathbb{P}$ is well-formed (see Definition III.2.10) and X contains no codimension $c+1$ singular stratum of $\mathbb{P}$.*

This means that any codimension 1 stratum of X is either nonsingular on $\mathbb{P}$, or an intersection $X \cap S$, where S is a codimension 1 stratum of $\mathbb{P}$.

III.2.21 Note.

(i) The hypersurface X_d in $\mathbb{P}(a_0, ..., a_n)$ is well-formed if

(1) $\operatorname{hcf}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n) | d$

(2) $\operatorname{hcf}(a_0, ..., \hat{a}_i, ..., a_n) = 1$

for all distinct i, j.

(ii) The codimension 2 weighted complete intersection X_{d_1, d_2} in $\mathbb{P}(a_0, ..., a_n)$ is well-formed if

(1) for all distinct i, j and k one of the following holds:

either

$$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_k, \ldots, \hat{a}_j, \ldots, a_n) | d_1$$

or $$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_k, \ldots, a_j, \ldots, a_n) | d_2$$

(2) for all distinct i and j then

$$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n) | d_1$$

and $$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_n) | d_2$$

(3) for all i $\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$

III.2.22 The adjunction formula. If $X_{d_1, \ldots, d_c}$ in $\mathbf{P}(a_0, \ldots, a_n)$ is well-formed and quasismooth then $\omega_X \cong \mathcal{O}_X(\sum d_i - \sum a_i)$ (see [WPS, Theorem 3.3.4]). This difference of sums will usually be denoted by α.

III.2.23 Note. The adjunction formula does not hold if the weighted complete intersection is not well-formed. For example consider the curve C_7 in $\mathbf{P}(1, 2, 3)$. Let $D \subset \mathbf{P}^2$ be the curve $\sigma^{-1}(C)$ where $\sigma : \mathbf{P}^2 \longrightarrow \mathbf{P}$ is the quotient map (see section III.2.11). By Hurwitz Theorem (see [Hart, Corollary IV.2.4]) we calculate that $g(C) = 1$ and so $\omega_C \cong \mathcal{O}_C$. This contradicts the adjunction formula since $\alpha = 1$.

From [WPS, Theorem 3.2.4(iii)] we have:

III.2.24 Lemma: *Let X be a well-formed quasismooth weighted projective complete intersection. Then*

$$\mathrm{H}^i(X, \mathcal{O}_X(n)) = 0$$

for all $n \in \mathbf{Z}$ and $i = 1, \ldots, \dim X - 1$.

In particular if S is a well-formed quasismooth weighted projective complete intersection of dimension 2 then the following are equivalent:

(i) S is a K3 surface.

(ii) $\omega_X \cong \mathcal{O}_S$.

(iii) $\alpha = 0$.

III.3 Quasismoothness

In this section we prove conditions for quasismoothness for hypersurfaces and codimension 2 weighted complete intersections. There are also some lemmas used in counting the number of intersections along 1 and 2 dimensional strata, along with a worked example of how to determine the number and types of singularity on a weighted hypersurface. First we consider the problem of quasismoothness.

III.3.1 Theorem: *The general hypersurface X_d in $\mathbf{P} = \mathbf{P}(a_0, \ldots, a_n)$ of degree d, where $n \geq 1$ is quasismooth if and only if*

either (1)

there exists a variable x_i for some i of weight d

or (2)

for every nonempty subset $I = \{i_0, \ldots, i_{k-1}\}$ of $\{0, \ldots, n\}$

either (a)

there exists a monomial $x_I^M = x_{i_0}^{m_0} \ldots x_{i_{k-1}}^{m_{k-1}}$ of degree d,

or (b)

for $\mu = 1, \ldots, k$, there exist monomials $x_I^{M_\mu} x_{e_\mu} = x_{i_0}^{m_{0,\mu}} \ldots x_{i_{k-1}}^{m_{k-1,\mu}} x_{e_\mu}$ of degree d, where $\{e_\mu\}$ are k distinct elements.

III.3.2 Note. If f can be written as $f = x_i + g$ for some x_i then X_d is clearly quasismooth. In this case X_d in $\mathbf{P}$ will be said to be a *linear cone*. So we need only consider the case where f is not linear in any of the variables (i.e. $\deg x_i = a_i \neq d$ for all i).

Proof. Assume that X_d in $\mathbf{P}$ is not a linear cone. Let F be the linear system of all homogeneous polynomials of degree d with respect to the weights $\{a_i\}$. Let $f \in F$ be a sufficiently general polynomial. Define $X_d : (f = 0) \subset \mathbf{P}$.

$$\begin{tikzcd} C_X^* \arrow[r, "i"] \arrow[d] & \mathbf{A}^{n+1} - \underline{0} \arrow[d] \\ X_d \arrow[r, "i"] & \mathbf{P} \end{tikzcd}$$

Note that the point $\underline{0}$ is a base point and is usually singular; as this point does not lie in C_X^* this does not affect quasismoothness. By Bertini's Theorem (see [Hart, Remark III.10.9.2]) the only singularities of the general C_X^* lie on the base locus of the linear system F. Any component of the base locus is just a coordinate k-plane for some $k = 0, \ldots, n$. So the general hypersurface X_d is quasismooth if and only if the general hypersurface C_X^* is nonsingular at each point of its intersection with every coordinate k-plane contained in the base locus.

Let Π be a coordinate k-plane for some $k = 1, \ldots, n$. By renumbering, assume that Π is given by $x_k = \ldots = x_n = 0$, corresponding to the subset $I = \{0, \ldots, k-1\}$. Let $\Pi^0 \subset \Pi$ be the open toric stratum where $x_0, \ldots, x_{k-1}$ are nonzero. Expand f in terms of the coordinates $x_k, \ldots, x_n$:

$$f = h(x_0, \ldots, x_{k-1}) + \sum_{i=k}^{n} x_i g_i(x_0, \ldots, x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, \ldots, x_n \end{array} \right\}.$$

Assume that one of Conditions (a) and (b) hold for I. If (a) holds (i.e. h is nonzero) then Π is not part of the base locus, and so by Bertini's Theorem Π^0 contains no singular points. Geometrically this means that C_X^* intersects Π^0 transversally and so Π^0 is normal to the hypersurface at the points of intersection.

Assume that only (b) holds. So $h = 0$ and $\Pi \subset C_X^*$. By (b) there are at least k of the $\{g_i\}$ which are nonzero. Singular points occur exactly on the locus $Z = \bigcap_i (g_i = 0) \subset \Pi^0$, which is an intersection of at least k free linear systems on Π^0. Thus $\dim Z \le 0$. As Z is a quasicone, it is at worst the origin (compare Lemma III.2.15). Therefore C_X^* is nonsingular along Π^0.

As one of these 2 conditions hold for every nonempty subsets I, C_X^* is nonsingular.

Conversely assume that Conditions (a) and (b) do not hold for all I. Let I be a subset for which these 2 conditions fail. Without loss of generality assume that $I = \{0, ..., k-1\}$. Let Π be the corresponding coordinate k-plane $x_k = ... = x_n = 0$. As (a) and (b) do not hold

$$f = \sum_{i=k}^{n} x_i g_i(x_0, ..., x_{k-1}) + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, ..., x_n \end{matrix} \right\}$$

and at most $k-1$ of the $\{g_i\}$ are nonzero.

As above singular points occur exactly on the intersection $Z = \bigcap_{i \ge k} (g_i = 0) \cap \Pi$. Since there are at most $k-1$ of the $\{g_i\}$ which are nonzero, $\dim Z \ge k - (k-1) = 1$. Thus Z is nonempty and so C_X^* is singular on Π.

Therefore Conditions (a) and (b) are both sufficient and necessary for quasismoothness when X_d in not a linear cone.

□

III.3.3 Note.

(i) The only quasismooth cones are the linear cones. Suppose a variable x_i does not occur in the defining equation f. So $C_X = C_{X'} \times \mathbf{A}^1$ where $X' : (f = 0) \subset \mathbf{P}(a_0, ..., \hat{a}_i, ..., a_n)$. Suppose that $C_{X'}$ has a singularity at the origin. Thus $C_{X'} \times \mathbf{A}^1$ has a line of singularities along $0 \times \mathbf{A}^1$; a contradiction. So $C_{X'}$ is nonsingular at the origin and so f must be linear in a variable; this is the linear cone case.

(ii) Without loss of generality we can assume in (b) that $e_\mu \in \{0, ..., n\} - I$, since otherwise this is Condition (a).

(iii) For $2|I| \ge n+1$ Condition (b) implies Condition (a), since there are simply not enough variables x_j.

(iv) Condition (b), with $|I| = 1$, of the theorem gives that for all $i = 0, ..., n$ there must exist a monomial $x_i^{m} x_{e_i}$, for some e_i, of degree d. This is equivalent to requiring that C_X^* is smooth along the coordinate axes (i.e. X_d is quasismooth at the vertices) and is in practice the most substantial case. Weighted hyperspaces (and polynomials) which satisfy this condition will be said to be *semi-quasismooth*.

(v) C_X contains no coordinate stratum of dimension $\ge (n+1)/2$ except possibly in the linear cone case.

So we have the following corollaries for curves, surfaces and 3-folds.

III.3.4 Corollary: *The curve C_d in $\mathbf{P}(a_0, a_1, a_2)$, where $d > a_i$, is quasismooth if and only if the following hold for all i:*

(1) there exists a monomial $x_i^n x_{e_i}$ for some e_i of degree d.

(2) there exists a monomial of degree d which does not involve x_i.

Proof. Since $d > a_i$ for all i, X_d is not a linear cone. Conditions (1) and (2) come from considering the conditions of the above theorem for $|I| = 1$ and $|I| = 2$ respectively. □

The proofs of the following corollaries are similar to the above.

III.3.5 Corollary: *The surface S_d in $\mathbf{P}(a_0, ..., a_3)$, where $d > a_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$ for some e_i of degree d.

(2) for all distinct i, j

either

there exists a monomial $x_i^m x_j^n$ of degree d,

or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.

(3) there exists a monomial of degree d which does not involve x_i.

III.3.6 Corollary: *The 3-fold X_d in $\mathbf{P}(a_0, ..., a_4)$, where $d > a_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$ of degree d.

(2) for all distinct i, j

either

there exists a monomial $x_i^m x_j^n$ of degree d,

or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.

(3) there exists a monomial of degree d which does not involve either x_i or x_j.

In the codimension 2 case we have:

III.3.7 Theorem: *Suppose the general codimension 2 weighted complete intersection X_{d_1, d_2} in $\mathbf{P} = \mathbf{P}(a_0, ..., a_n)$, where $n \geq 2$, of multidegree $\{d_1, d_2\}$ is not the intersection of a linear cone with another hypersurface. X_{d_1, d_2} in $\mathbf{P}$ is quasismooth if and only if for each nonempty subset $I = \{i_0, ..., i_{k-1}\}$ of $\{0, ..., n\}$ one of the following holds:*

(a) there exists a monomial $x_I^{M_1}$ of degree d_1 and there exists a monomial $x_I^{M_2}$ of degree d_2

(b) there exists a monomial x_I^M of degree d_1, and for $\mu = 1, ..., k-1$ there exist monomials $x_I^{M_\mu} x_{e_\mu}$ of degree d_2, where $\{e_\mu\}$ are $k-1$ distinct elements.

(*c*) *there exists a monomial x_I^{M} of degree d_2, and for $\mu = 1, \ldots, k-1$ there exist monomials $x_I^{M_\mu} x_{e_\mu}$ of degree d_1, where $\{e_\mu\}$ are $k-1$ distinct elements.*

(*d*) *for $\mu = 1, \ldots, k$, there exist monomials $x_I^{M^1_\mu} x_{e^1_\mu}$ of degree d_1, and $x_I^{M^2_\mu} x_{e^2_\mu}$ of degree d_1, such that $\{e^1_\mu\}$ are k distinct elements, $\{e^2_\mu\}$ are k distinct elements and $\{e^1_\mu, e^2_\mu\}$ contains at least $k+1$ distinct elements.*

Proof. Let F_1 and F_2 be linear systems of all homogeneous polynomials of degrees d_1 and d_2 respectively with respect to the weights $a_0, \ldots, a_n$. Let $f_1 \in F_1$ and $f_2 \in F_2$ be sufficiently general polynomials. Define

$$X = X_{d_1, d_2} : (f_1 = 0, f_2 = 0) \subset \mathbf{P}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} C_X & \xrightarrow{i} & \mathbf{A}^{n+1} - \underline{0} \\ \Big\downarrow & & \Big\downarrow \\ X & \xrightarrow{i} & \mathbf{P} \end{array}$$

The only singularities that can occur in the general member of the family occur on the coordinate strata. So as in the proof of quasismoothness for hypersurfaces, X is quasismooth if and only if C_X is smooth along all the coordinate strata.

Assume that one of Conditions (*a*), (*b*), (*c*) or (*d*) hold for each nonempty subset I. Let Π be a coordinate k-plane for some k. By renumbering, we can assume that Π is given by $x_k = \ldots = x_n = 0$, corresponding to the subset $I = \{0, \ldots, k-1\}$. As before let Π^0 be the open toric strata where $x_0, \ldots, x_{k-1}$ are nonzero. Expand both f_1 and f_2 in terms of the coordinates $x_k, \ldots, x_n$:

$$f_\lambda = h_\lambda(x_0, \ldots, x_{k-1}) + \sum_{i=k}^{n} x_i g^i_\lambda(x_0, \ldots, x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, \ldots, x_n \end{array} \right\}$$

for $\lambda = 1, 2$.

Suppose (*a*) holds. So h_1 and h_2 are nonzero on Π^0. If either h_1 or h_2 involves only one monomial then $\Pi^0 \cap C_X$ is empty. This includes the case when $k = 1$. So without loss of generality assume that h_1 and h_2 each involve at least 2 monomials and hence $k \geq 2$. Π^0 is not part of the base locus of F_1 or F_2. By Bertini's Theorem ($f_1 = 0$) and ($f_2 = 0$) are nonsingular on Π^0. Since ($h_1 = 0$) and ($h_2 = 0$) are free linear systems on Π^0, ($h_1 = 0$) and ($h_2 = 0$) intersect transversally. Thus, at each point of $(h_1 = h_2 = 0) \cap \Pi^0$, there exist 2 distinct normals. Therefore C_X is nonsingular along Π^0.

Suppose (*b*) holds. So h_1 is nonzero and there are at least $k-1$ of the $\{g^i_1\}$ which are nonzero. So Π^0 is not part of the base locus for F_1, and so by Bertini's Theorem we have that ($f_1 = 0$) is nonsingular on Π^0. Singular points occur exactly on the locus $Z = (h_1 = 0) \cap_i (g^i_2 = 0) \subset \Pi^0$, which is an intersection of at least $k-1$ free linear systems on $(h_1 = 0) \cap \Pi^0$. Thus $\dim Z \leq 0$ and hence is at worst the origin. Therefore C_X^I is nonsingular along Π^0.

The case where Condition (c) holds is similar to Condition (b).

Suppose that only Condition (d) holds. We have

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \dots, x_{k-1}) + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, \dots, x_n \end{matrix} \right\}$$

for $\lambda = 1, 2$. The normal directions, perpendicular to the plane Π, to the hypersurfaces are $(g_1^k, \dots, g_1^n)$ and $(g_2^k, \dots, g_2^n)$. Define the matrix M_P by

$$M_P = \begin{bmatrix} g_1^k(P) & \dots & g_1^n(P) \\ g_2^k(P) & \dots & g_2^n(P) \end{bmatrix}.$$

Singular points occur exactly on the locus $Z = \{P : \operatorname{rank} M_P \leq 1\}$. As there are at least k monomials of the form $x_i^M x_e$ of degree d_λ, at least k of the $\{g_\lambda^i\}$ are nonzero. As these are free on Π^0, each row of the matrix M_P is nonzero for each $P \in \Pi^0$. Furthermore this matrix for any $P \in Z$ has at least $k+1$ nonzero columns, since there are at least $k+1$ distinct elements in $\{e_\mu^1, e_\mu^2\}$. By renumbering we can assume that the first $k+1$ columns of M^P are not identically zero on Π^0.

Fix $P \in \Pi^0$. Without loss of generality we can assume that $g_1^k(P) \neq 0$. If $g_2^k(P) = 0$ then $g_2^i(P) \neq 0$ for some $i > k$, and so M^P has rank 2. In this case $P \in C_X^I$ is nonsingular. Suppose that $g_2^k(P) \neq 0$. Define $a = g_1^k(P)$, $b = g_2^k(P)$ and

$$Z_P = \cap_{i>k}(a g_2^i(Q) - b g_1^i(Q) = 0) \subset \Pi^0.$$

Notice that $P \in Z_P$ if and only if $\operatorname{rank} M_P \leq 1$, which is equivalent to $P \in C_X^I$ being singular. Since Z_P is the intersection of k free linear systems on Π^0, $\dim Z_P \leq 0$ and so Z_P is at worst the origin. In particular $P \notin Z_P$ and hence $P \in C_X^I$ is nonsingular. Therefore C_X^I is nonsingular along Π^0.

As one of these 4 conditions hold for every nonempty subsets I, C_X^I is nonsingular.

Conversely assume that none of the conditions (a), (b), (c) or (d) hold for some nonempty subset I. Without loss of generality we can assume that $I = \{0, \dots, k-1\}$. Let Π be the corresponding coordinate plane $x_k = \dots = x_n = 0$. There are 3 cases:

(i) $\Pi \not\subset C_{X_{d_1}}$

So h_1 is nonzero and there are at most $k-2$ of the $\{g_2^i\}$ which are nonzero. The singular points are exactly the locus $Z = (h_1 = 0) \cap_i (g_2^i = 0)$. But $\dim Z \geq k - (k-2) - 1 = 1$ and so Z contains more than the origin. Thus C_X^I is singular along Π.

(ii) $\Pi \not\subset C_{X_{d_2}}$

Similarly in this case C_X^I is singular along Π.

(iii) $\Pi \subset C_{X_{d_1}} \cap C_{X_{d_2}}$

In this case both h_λ are identically zero. So

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \dots, x_{k-1}) + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, \dots, x_n \end{matrix} \right\}$$

for $\lambda = 1, 2$. As Condition (d) does not hold one of 2 cases occurs:

either (1)

for some λ there are at most $k-1$ of the $\{g_\lambda^i\}$ which are nonzero. Thus $Z_\lambda = \bigcap_i (g_\lambda^i = 0)$ has dimension at least 1 and so these $\{g_\lambda^i\}$ have a common solution. Therefore the matrix

$$M_P = \begin{bmatrix} g_1^1(P) & \dots & g_1^n(P) \\ g_2^1(P) & \dots & g_2^n(P) \end{bmatrix}$$

has rank less than 2 for some $P \in Z_\lambda$ and hence C_I is singular along Π.

or (2)

there are at most k distinct elements in $\{e_\mu^1, e_\mu^2\}$. Thus there are at most k nonzero columns in the matrix M_P. Let $Z = \{P : \operatorname{rank} M_P \leq 1\}$. Thus $\dim Z \geq k - (k-1) = 1$ and so contains more than just the origin. Therefore C_I is singular along Π.

So if one of these 4 conditions are not satisfied for every subset I then C_I is singular.

$\square$

III.3.8 Corollary: *Suppose X_{d_1, d_2} in $\mathbf{P}$ is quasismooth and is not the intersection of a linear cone with another hypersurface. We have the following:*

(i) Every variable x_i occurs in at least one of the defining equations.

(ii) All but at most one variable are in both equations.

(iii) If x_i does not appear in one of the defining equations then there exists a monomial x_i^m occurring in the other equation.

Proof.

(i) This follows from the previous theorem with $|I| = 1$.

(ii) Suppose, after renumbering, that x_0 and x_1 were not involved in f_1. Then none of the conditions can hold for $I = \{0, 1\}$, a contradiction.

(iii) Suppose that x_i does not appear in f_1. Conditions (a), (b) and (d) cannot hold and so there must be a monomial x_i^m of degree d_2. Geometrically if one of the hypersurfaces is a singular along a coordinate axis, because the equation f does not involve that variable, then the other hypersurface cannot pass through that axis.

$\square$

We now reduce cyclic isolated canonical singularities to combinatorial conditions.

III.3.9 Lemma: *A canonical curve point is smooth.*

This is clear since canonical singularities are normal. For dimension 2 we have:

III.3.10 Lemma: *The following are equivalent:*

(i) Q in S is a cyclic quotient canonical surface singularity.

(2) *Q is of type* $\frac{1}{r}(a, -a)$ *for some index r and a coprime to r.*

(3) *Q is of type* $\frac{1}{r}(1, -1)$ *for some index r.*

The above singularities are Du Val singularities of type A_{r-1}. For 3-folds we have:

III.3.11 Lemma: *The following are equivalent:*

(1) *S is an isolated terminal 3-fold quotient singularity.*

(2) *S is of type* $\frac{1}{r}(b_0, b_1, b_2)$, *for some positive integers* r, b_0, b_1, b_2, *with* $r \geq 2$, *r and* b_i *coprime and* $r | b_i + b_j$ *for a pair of distinct i, j.*

(3) *S is of the form* $\frac{1}{r}(1, -1, b)$ *for some* $r \geq 2$ *and b coprime to r.*

Finally, the following 2 lemmas are very useful for calculating the number and arrangement of singularities on a complete intersection.

III.3.12 Lemma: *Let x and y be of weight* a_0 *and* a_1 *respectively, where* $\mathrm{hcf}(a_0, a_1) = 1$. *Suppose* $f(x, y)$ *is homogeneous polynomial of degree d, semi-quasismooth (see Note III.3.3(iv)) and sufficiently general. Then* $X_d : (f = 0)$ *in* $\mathbb{P}(a_0, a_1)$ *is a finite set and:*

(i) P_i *is in* X_d *if and only if* $a_i \nmid d$ *for* $i = 0, 1$,

(ii) *there are exactly* $\left\lfloor \frac{d}{a_0 a_1} \right\rfloor$ *other points in* X_d.

Proof. There are 4 cases:

(i) $a_0 | d, a_1 | d$.

Then f is of the form

$$f = x^{d/a_0} + \ldots + y^{d/a_1},$$

written using the coefficient convention (see III.2.19). So

$$\frac{f}{y^{d/a_1}} = \left[\frac{x^{a_1}}{y^{a_0}}\right]^{d/a_0 a_1} + \ldots + 1,$$

which has exactly $\frac{d}{a_0 a_1}$ roots.

(ii) $a_0 \nmid d, a_1 | d$.

Since X_d is semi-quasismooth, f is of the form

$$f = y(x^{(d-a_1)/a_0} + \ldots + y^{(d-a_1)/a_1}).$$

The solution $y = 0$ gives the point P_0.

$$\frac{f}{y^{d/a_1}} = \left[\frac{x^{a_1}}{y^{a_0}}\right]^{(d-a_1)/a_0 a_1} + \ldots + 1.$$

This has exactly $n = \frac{d-a_1}{a_0 a_1}$ roots. So $d = n a_0 a_1 + a_1$. As $a_0 \nmid d$ then $a_0 > 1$, and so

$a_1 < a_0 a_1$. Thus $n = \left\lfloor \dfrac{d}{a_0 a_1} \right\rfloor$.

(iii) $a_0 | d$, $a_1 \nmid d$.

Similar to (ii).

(iv) $a_0 \nmid d$, $a_1 \nmid d$.

$$f = xy(x^{(d-a_0-a_1)/a_0} + \ldots + y^{(d-a_0-a_1)/a_1})$$

So the 2 vertices P_0 and P_1 are solutions. Also

$$\frac{f}{xy^{d/a_1}} = \left(\frac{x^{a_1}}{y^{a_0}}\right)^{(d-a_0-a_1)/a_0 a_1} + \ldots + 1,$$

which has exactly $n = \dfrac{d - a_0 - a_1}{a_0 a_1}$ roots on $\mathbf{P} - \{P_0, P_1\}$. So $d = n a_0 a_1 + (a_0 + a_1)$. As $a_0 \nmid d$ and $a_1 \nmid d$ then $a_0, a_1 \geq 2$ and not both equal to 2. Thus

$$a_0 a_1 = (a_0 - 1)(a_1 - 1) - 1 + a_0 + a_1 > a_0 + a_1.$$

Therefore $n = \left\lfloor \dfrac{d}{a_0 a_1} \right\rfloor$.

□

III.3.13 Lemma: *Let x_0, x_1 and x_2 have weights a_0, a_1 and a_2, where* $\mathrm{hcf}(a_0, a_1, a_2) = 1$. *Suppose f and g are sufficiently general semi-quasismooth homogeneous polynomials in $\mathbf{k}[x_0, x_1, x_2]$ of degrees d and e respectively. Suppose that $X_{d,e} : (f = 0, g = 0)$ in $\mathbf{P}(a_0, a_1, a_2)$ is a finite set. Let*

$n_{i,j}$ be the number of points of $X_{d,e}$ along the edge $P_i P_j$,

$h_{i,j} = \mathrm{hcf}(a_i, a_j)$,

n_i be the number of points at the vertex P_i (i.e. $n_i = 0, 1$),

N be the number of points in $\mathbf{P} - \Delta$.

Then:

$$\frac{de}{a_0 a_1 a_2} = \sum_i \frac{n_i}{a_i} + \sum_{i>j} \frac{n_{i,j}}{h_{i,j}} + N$$

III.3.14 Note.

(1) $X_{d,e}$ in $\mathbf{P}$ is not automatically finite (consider $X_{5,9}$ in $\mathbf{P}(1, 2, 4)$).

(2) Similar results hold for higher codimensions and involve use of induction on the dimension.

(3) Notice that Lemma III.3.12 can be deduced from the above (consider $X_{d,1}$ in $\mathbf{P}(a_0, a_1, 1)$).

(4) This also has connections with the Minkowski mixed volumes of Newton polyhedra (see III.3.18).

Proof. Let $\sigma : \mathbb{P}^2 \longrightarrow \mathbb{P}$ be the quotient map defined in III.2.11. Let $F = \sigma^* f$ and $G = \sigma^* g$. By Bézout's theorem $Y = V(F, G)$ in $\mathbb{P}^2$ consists of exactly de points counted with multiplicity.

The restriction of σ to $\mathbb{P}^2 - \Delta$ is $a_0 a_1 a_2$ to 1, onto $\mathbb{P} - \Delta$. As there are N points on $\mathbb{P} - \Delta$ this accounts for $a_0 a_1 a_2 N$ points on $\mathbb{P}^2 - \Delta$.

The restriction of σ to the line $Q_i Q_j$ is $a_i a_j / h_{i,j}$ to 1, onto $P_i P_j$. Without loss of generality assume that $h_{i,j} | d$ but that $h_{i,j} \nmid e$. Notice that $x_k | g$, or else there would exist a monomial $x_i^a x_j^b$ of degree e, contradicting $h_{i,j} \nmid e$. Then f and g are of the form:

$$f = x_i^m x_j + x_j^m x_i + \dots$$
$$g = x_k (x_i^{n'} + x_j^{m'} + \dots)$$

Thus F and G are of the form:

$$F = X_i^{m a_i} X_j^{a_j} + X_j^{m a_j} X_i^{a_i} + \dots$$
$$G = X_k^{a_k} (X_i^{n' a_i} + X_j^{m' a_j} + \dots)$$

Localise by setting $X_i = 1$, to give corresponding affine equations $\overline{F}$, $\overline{G}$. Let $[X_i, X_j, X_k] = [1, \xi, 0]$ be a point of intersection along the line $Q_i Q_j$. The multiplicity μ of intersection is given by:

$$\begin{aligned}
\mu &= \mathrm{mult}(F, G, [1, \xi, 0]) \\
&= \mathrm{mult}(\overline{F}, \overline{G}, (\xi, 0)) \\
&= \mathrm{mult}(X_j^{a_j} + X_i^{m a_i} + \dots, X_k^{a_k}, (\xi, 0)) \\
&= \mathrm{mult}(X_j' + \dots, X_k^{a_k}, (0, 0)) \\
&= a_k
\end{aligned}$$

where $X_j' = X_j - \xi$. So this line contributes $n_{i,j} a_k / h_{i,j}$ points (counted with multiplicity) to Bézout's theorem.

Consider the vertex Q_i. If P_i is contained in X then $a_i \nmid d$ and $a_i \nmid e$. As X is semi-quasismooth, $a_i | d - a_j$ and $a_i | e - a_k$ for distinct i, j, and k. So f and g are of the form:

$$f = x_i^n x_j + \dots$$
$$g = x_i^m x_k + \dots$$

Thus:

$$F = X_i^{n a_i} X_j + \dots$$
$$G = X_i^{m a_i} X_k + \dots$$

The intersection multiplicity μ at Q_i is:

$$\mu = \mathrm{mult}(F, G, Q_i)$$

Localising at $X_i = 1$ gives:

$$\begin{aligned}
\mu &= \mathrm{mult}(\overline{F}, \overline{G}, (0, 0)) \\
&= \mathrm{mult}(X_j^{a_j} + \dots, X_k^{a_k} + \dots, (0, 0))
\end{aligned}$$

$$= a_j a_k.$$

Clearly $X_j^{a_j}$ and $X_k^{a_k}$ are the smallest degree monomials in $\overline{F}$ and $\overline{G}$. So this gives a contribution of $a_j a_k n_i$.

Combining the above gives:

$$de = \sum_{\text{distinct } i,\, j,\, k} n_i a_j a_k + \sum_{i>j} \frac{n_{i,j} a_i a_j a_k}{n_{i,j}} + N$$

which rearranges to give the formula in the lemma.

□

An alternative proof of the above two lemmas is via Newton polyhedra and the Minkowski mixed volume (see both [Be] and [Ku]).

III.3.15 Definition: *An integral polyhedron S is a polyhedron in $\mathbf{R}^n$ with vertices in $\mathbf{Z}^n$. The n-dimensional volume of S will be denoted by $V_n(S)$, where the volume of the unit parallelepiped is 1.*

III.3.16 Definition: *Let $f \in \mathbf{k}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ be a Laurent polynomial. For each $m = (m_1, ..., m_n) \in \mathbf{Z}^n$ define $x^m = x_1^{m_1} \ldots x_n^{m_n}$. Then*

$$f = \sum_{m \in \mathbf{Z}^n} c_m x^m.$$

The Newton polyhedron Newton(f) *of f is the convex hull of $\{m \in \mathbf{Z}^n : c_m \neq 0\}$, and is an integral polyhedron.*

III.3.17 Definition: *Let $S = \{S_i : i = 1, ..., n\}$ be a set of integral polyhedra. The Minkowski mixed volume $V(S)$ of S is given by:*

$$V(S) = (-1)^{n-1} \sum V_n(S_i) + (-1)^{n-2} \sum_{i<j} V_n(S_i + S_j) + \ldots + V_n(S_1 + \ldots + S_n)$$

where $S_i + S_j = \{s_i + s_j : s_i \in S_i, s_j \in S_j\}$. This is the classical formula up to a multiple of $n!$

Let T^n be the n-dimensional torus $(\mathbf{k}^*)^n$. This corresponds to the open toric stratum in $\mathbf{P}$. Let $\mathcal{F}$ be a system of n sufficiently general Laurent polynomials $\{f_i\}$ with corresponding Newton polyhedra $S = \{S_i\}$. The roots of these n polynomials are isolated. Let $L(\mathcal{F})$ be the number counted with multiplicity of such roots. Then [Be, Theorem A] gives:

$$L(\mathcal{F}) = V(S).$$

III.3.18 Alternative proof of Lemma III.3.12. Let T^1 be the torus $x_0 x_1 \neq 0$ in $\mathbf{P} = \mathbf{P}(a_0, a_1)$. Suppose that $a_0, a_1 | d$. Then $f = x_0^{d/a_0} + \ldots + x_1^{d/a_1}$. So

$$N_f = \text{Newton}(f) = [(d/a_0, 0), (0, d/a_1)],$$

where $[P, Q]$ denotes the line segment from P to Q. So $V_1(N_f) + 1$ is the number of integral points on N_f, i.e. the number of solutions to

$$\{(\alpha, \beta) \in \mathbf{Z}^2 : \alpha \geq 0, \beta \geq 0, \alpha a_0 + \beta a_1 = d\}.$$

For a solution (α, β) we have $\alpha = (d - \beta a_1)/a_0 \in \mathbf{Z}$, i.e. $d \equiv \beta a_1 \bmod a_1$. As a_0 and a_1 are coprime, then a_1 is invertible modulo a_0, with inverse s. So $\beta \equiv ds \bmod a_0$, i.e. $\beta = ds + na_0$ for some n. Also $0 \leq \beta \leq d/a_1$. So

$$-\frac{ds}{a_0} \leq n \leq \frac{d}{a_0 a_1} - \frac{ds}{a_0}.$$

There are $\dfrac{d}{a_0 a_1} + 1$ such solutions. Thus f has $\dfrac{d}{a_0 a_1}$ roots on the torus T^1 in $\mathbf{P}$.

Similarly when $a_0 \nmid d$, etc..

$\square$

Lemma III.3.13 can be proved using analogous methods.

III.3.19 Determination of singularities on a weighted hypersurface.

In this section I shall work through the calculation of how to determine the singularities of X_{46} in $\mathbf{P}(4, 5, 6, 7, 23)$. Let v, w, x, y and z be the homogeneous coordinates of $\mathbf{P} = \mathbf{P}(4, 5, 6, 7, 23)$ of weights 4, 5, 6, 7 and 23 respectively. Let f be a general polynomial of homogeneous degree 46. Then f (using the coefficient convention) is of the form:

$$f = v^{10}x + w^8x + x^7v + y^6v + z^2 + \text{others}.$$

This is well-formed and quasismooth (see Lemma III.3.1). So the singularities of the hypersurface occurs only on the edges and at the vertices of $\mathbf{P}$.

Consider each of the vertices in reverse order:

P_4: Since f contains the monomial z^2 with a non-zero coefficient, $f(P_4) \neq 0$ and so $P_4 \notin X_{46}$.

P_3: There is no monomial of the form y^n for any n in f, and so $P_3 \in X_{46}$. Consider the affine piece $(y = 1)$. $P_3 \in X_{46}$ looks like:

$$(\tilde{f} = f(v, w, x, 1, z) = v + \ldots = 0) \subset \mathbf{A}^4 / \langle\varepsilon\rangle$$

where ε is a primitive 7th root of unity and acts as:

$$v \longmapsto \varepsilon^4 v$$
$$w \longmapsto \varepsilon^5 w$$
$$x \longmapsto \varepsilon^6 x$$
$$z \longmapsto \varepsilon^{23} z$$

Since $\partial f/\partial v = y^6 + \ldots$ is non-zero at P_3. By the Inverse Function Theorem w, x and z are local coordinates on X_{46} around $P_3 \in X_{46}$. Thus the singularity here is of type $\frac{1}{7}(5, 6, 23)$. This is equivalent to $\frac{1}{7}(6, 1, 3)$, which is terminal.

P_2: Again there is no monomial of the form x^n for any n in f, and so $P_2 \in X_{46}$. Consider the affine piece $(x = 1)$. $P_2 \in X_{46}$ looks like:

$$(\tilde{f} = f(v, w, 1, y, z) = v + \ldots = 0) \subset \mathbf{A}^4 / \langle\varepsilon\rangle$$

where ε is a primitive 6^{th} root of unity and acts as:

$$v \longmapsto \varepsilon^4 v$$

$$w \mapsto \varepsilon^5 w$$
$$y \mapsto \varepsilon y$$
$$z \mapsto \varepsilon^{23} z$$

So $\partial f/\partial v = x^7 + \ldots$ is non-zero at P_3. By the Inverse Function Theorem, w, y and z are local coordinates on X_{46} around $P_2 \in X_{46}$. Thus the singularity here is of type $\frac{1}{6}(5, 7, 23)$. This is equivalent to $\frac{1}{6}(5, 1, 1)$, which is terminal.

P_1: $P_1 \in X_{46}$ is locally $f = x + \ldots = 0$ and gives a terminal singularity of type $\frac{1}{5}(4, 1, 2)$.

P_0: $P_0 \in X_{46}$ is locally $f = x + \ldots = 0$ and gives a terminal singularity of type $\frac{1}{4}(3, 1, 1)$.

Consider the edges of $\mathbf{P}$. An edge $P_i P_j$ is singular if and only if $h = \mathrm{hcf}(a_i, a_j) \neq 1$. In which case it is analytically equivalent to $\mathbf{k}^* \times \frac{1}{h}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_4)$. So only the edge $P_0 P_2$ is singular and looks like $\mathbf{k}^* \times \frac{1}{2}(1, 1, 1)$. Since $2 = \mathrm{hcf}(4, 6) | 46$, the hypersurface does not contain this line. Lemma III.3.12 is used on X_{46} in $\mathbf{P}(4, 6)$, after cancelling the common factor, to give 3 points of intersection. Alternatively,

$$f_{|P_0P_2} = uxg_{36}(u, x) = uxg_3(u^3, x^2),$$

where g_{36} and g_3 are polynomials of degree 36 and 3 respectively. There are exactly 3 solutions to $g_3 = 0$, and so there are 3 points of intersection. So X_{46} crosses $P_0 P_4$ transversally and hence there are 3 singularities of type $\frac{1}{2}(1, 1, 1)$ along $P_0 P_2$.

Thus the hypersurface X_{46} in $\mathbf{P}(4, 5, 6, 7, 23)$ has the following singularities:

3 of type $\frac{1}{2}(1, 1, 1)$,
1 of type $\frac{1}{4}(3, 1, 1)$,
1 of type $\frac{1}{5}(4, 1, 2)$,
1 of type $\frac{1}{6}(5, 1, 1)$,
and 1 of type $\frac{1}{7}(6, 1, 3)$.

III.4 Weighted curve hypersurfaces.

III.4.1 Theorem: *A weighted curve complete intersection is smooth if and only if it is quasismooth.*

III.4.2 Theorem: *A weighted curve C_d in $\mathbf{P}(a_0, a_1, a_2)$ is well formed, not a linear cone and quasismooth if and only if for each i the following 4 conditions hold:*

(1) $a_i < d$,

(2) $a_i | d$,

(3) there exists a monomial of degree d which does not involve x_i,

and (4)

$\mathrm{hcf}(a_i, a_j) = 1$ *for all distinct i, j.*

Proof. C is well-formed if and only if $a_i | d$ for all i and $\mathrm{hcf}(a_i, a_j) = 1$ for all distinct i, j (see Note III.2.21 (i)). These are Conditions (2) and (4).

Suppose C is not a linear cone and quasismooth. Then Conditions (1) and (3) hold. Also $a_i | d - a_e$ for some e. But this already satisfied by Condition (2).

The converse follows immediately from Conditions (1), (2) and (3).

□

III.4.3 Smooth weighted curve hypersurfaces with $\alpha = 0$.

We list the only smooth weighted hypersurfaces of dimension 1 with $\alpha = 0$ satisfying the above conditions and $\sum a_i \leq 100$.

Curve	D
C_3 in $\mathbb{P}(1, 1, 1)$	$3P$
C_4 in $\mathbb{P}(1, 1, 2)$	$2P$
C_6 in $\mathbb{P}(1, 2, 3)$	P

All are elliptic curves (i.e. $g = 1$ and $\omega \cong \mathcal{O}_C$) and are given by Proj R where R is:

$$R = \bigoplus_{n \geq 0} \mathrm{H}^0(\mathcal{O}_C(D)),$$

and D is given in the above table.

III.4.4 Example. Consider an elliptic curve C and the divisor $D = 2P$, where P is any point on C. By Riemann-Roch,

$$h^0(nD) - h^1(nD) = \deg(nD) + (1 - g).$$

As $D > K \equiv 0$, then $h^1(nD) = 0$ for all $n \geq 1$. Also $g = 1$ and so

$$\begin{aligned} h^0(nD) &= \deg(nD) \\ &= 2n. \end{aligned}$$

Thus $h^0(D) = 2$ and $h^0(2D) = 4$. Let $\{x_0, x_1\}$ be a basis for $\mathrm{H}^0(D)$. Then x_0^2, $x_0 x_1$ and x_1^2 are linearly independent elements of $\mathrm{H}^0(2D)$. As $h^0(2D) = 4$ then there exists an extra element y of degree 4.

Consider the map:

$$\phi_n : \mathrm{H}^0(D) \otimes \mathrm{H}^0((n-1)D) \longrightarrow \mathrm{H}^0(nD).$$

Notice that x_0 and x_1 have no common base points. By the base-point-free pencil trick (see [ACGH, p. 126]),

$$\mathrm{Ker}\, \phi_n \cong \mathrm{H}^0((n-1)D - D) = \mathrm{H}^0((n-2)D)$$

which has dimension $2(n-2)$. Also $\mathrm{H}^0(D) \otimes \mathrm{H}^0(nD)$ has dimension $2.2(n-1)$. So $\dim \mathrm{Im}\, \phi_n = 2n$, and hence ϕ_n is onto for all $n \geq 2$. This means that $\mathrm{H}^0(nD)$ is generated from $\mathrm{H}^0(D)$ and $\mathrm{H}^0((n-1)D)$.

So we have the following table of bases for the $\mathrm{H}^0(nD)$.

n	$h^0(nD)$	monomials
1	2	x_0, x_1.
2	4	x_0^2, x_0x_1, x_1^2, y.
3	6	$x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0y, x_1y$.
4	8	$x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4, x_0^2y, x_0x_1y, x_1^2y, y^2$.

Notice that $H^0(4D)$ has dimension 8, but there are 9 monomials. Since ϕ_4 is onto then the first 8 in the list are linear independent. So there must be a relation of the form:

$$f = y^2 - g_4(x_0, x_1) + yh_2(x_0, x_1),$$

where g_4 and h_2 are homogeneous polynomials of degrees 4 and 2 respectively.

The number N_n of monomials in $H^0(nD)$ is given by:

$$N_n = 1 + n + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Suppose that f was the only relation, then the dimension of the module generated by the monomials of degree n is $N_n - 1.N_{n-4} = 2n$, which is the same as $h^0(nD)$.

So the ring R is $\mathbf{k}[x_0, x_1, y] / (f)$, where x_i has weight 1 and y has weight 2, i.e. the curve is C_4 in $\mathbf{P}(1, 1, 2)$. This technique should be compared to that in [M, Lecture 1, p. 17 - 21] and to Weierstrass normal form.

III.4.5 Smooth weighted curve hypersurfaces with $\alpha = 1$.

There are only 2 such curves which satisfy the conditions of Theorem III.4.2 and $\sum a_i \leq 100$:

curve	genus	ω_C
C_4 in $\mathbf{P}(1, 1, 1)$	3	$\mathcal{O}_C(1)$
C_6 in $\mathbf{P}(1, 1, 3)$	2	$\mathcal{O}_C(1)$

III.5 Weighted surface complete intersections.

In this section we give necessary and sufficient conditions for surface weighted complete intersections of codimension 1 and 2 to be quasismooth, well-formed and have at worst canonical singularities. We also include lists of such intersections.

III.5.1 Theorem: *Let S_d in $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ be a general hypersurface of degree d and let $\alpha = d - \sum a_i$. S_d is quasi-smooth, well-formed with at worst canonical quotient singularities and is not a linear cone if and only if all the following hold:*

(I) For all i,

(i) $d > a_i$.

(*ii*) *there exists e such that* $a_i|d-a_e$ (*i.e. there exists a monomial* $x_i^n x_e$ *of degree d*).

(*iii*) *there exists a monomial of degree d which does not involve* x_i.

(*iv*) *if* $a_i \nmid d$, *then* $a_i|\alpha$.

(2) *For all distinct i, j with* $h = \mathrm{hcf}(a_i, a_j)$

(*i*) *then* $h|d$.

(*ii*) $h|\alpha$.

(*iii*) *one of the following holds:*

either

there exists a monomial $x_i^m x_j^n$ *of degree d*,

or *there exist monomials* $x_i^{n_1}x_j^{m_1}x_{e_1}$ *and* $x_i^{n_2}x_j^{m_2}x_{e_2}$ *of degree d such that* e_1 *and* e_2 *are distinct.*

(3) *For all distinct i, j, k,* $\mathrm{hcf}(a_i, a_j, a_k) = 1$.

III.5.2 Note. Since the hypersurface is well-formed then $\omega_S = \mathcal{O}_S(\alpha)$.

Proof. Let f be a general homogeneous polynomial of degree d in variables $x_0, ..., x_3$; define $S_d : (f = 0) \subset \mathbb{P}$.

S_d is quasismooth and not a linear cone if and only if Conditions (1*i*), (1*ii*), (1*iii*) and (2*iii*) hold (see Corollary III.3.5).

Suppose furthermore that Conditions (1*iv*), (2*i*), (2*ii*) and (3) hold. As S_d is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities on the fundamental simplex $\Delta \subset \mathbb{P}$. By Condition (3) only vertices and edges need be checked.

Consider $P_i \in S_d$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d$. Condition (1*ii*) gives that there exists an $e \neq 0$ such that $a_0|d-a_e$. Without loss of generality we can assume that $e = 1$. So f is of the form $f = x_0^n x_1 + ...$ Thus $\partial f/\partial x_1$ is nonzero at P_0. By the Inverse Function Theorem x_2 and x_3 are local coordinates. So $P_0 \in S_d$ is of type $\frac{1}{a_0}(a_2, a_3)$. However $d = a_0 + ... + a_3 + \alpha$ and so $a_0|a_2 + a_3 + \alpha$. By Condition (1*iv*), $a_0|a_2 + a_3$. Let $h = \mathrm{hcf}(a_0, a_2)$. So $h|a_3$ and hence, by Condition (3), $h = 1$. Therefore $P_0 \in S_d$ is a canonical singularity.

Consider the edge P_iP_j. Again by renumbering assume that $i = 0$ and $j = 1$. f restricted to P_0P_1 is:

$$f = \sum x_0^n x_1^m$$

where the sum is taken over the set $\{(n, m) : na_0 + ma_1 = d\}$. If $a_0 \nmid d$ then $a_0|d-a_e$ for some $e \neq 0$. If $e \neq 1$ then $h = \mathrm{hcf}(a_0, a_1)|a_e$ and by Condition (4) $h = 1$. Then P_0P_1 is non-singular. So assume that either $a_0|d$ or $a_0|d-a_1$. Hence f is not identically zero on P_0P_1, and so $S_d \cap P_0P_1$ is finite. Each point in this intersection is of type $\frac{1}{h}(a_2, a_3)$. Since $d = a_0 + ... + a_3 + \alpha$ and $h|\alpha$ then $h|a_2 + a_3$. Also $\mathrm{hcf}(h, a_2) = 1$. Thus each point is canonical.

Therefore S_d in $\mathbb{P}$ has at worst canonical singularities.

Conversely assume that S_d is quasismooth, well-formed, not a linear cone and has at worst only canonical singularities. Suppose $a_i \nmid d$. By renumbering we can assume that $i = 0$. So $P_0 \in S_d$ and $a_0 | d - a_e$ for some e. Without loss of generality assume that $e = 1$. As above the singularity at $P_0 \in S_d$ is of type $\frac{1}{a_0}(a_2, a_3)$. Since this is canonical we have $a_0 | a_2 + a_3$ and so $a_0 | \alpha$. This is Condition (1*iv*).

Suppose $h = \mathrm{hcf}(a_i, a_j)$. By renumbering assume that $i = 0$ and $j = 1$. As S_d is well-formed then $h | d$, which is Condition (2*i*). So $P_0P_1 \cap S_d$ is a finite intersection, where each point is of type $\frac{1}{h}(a_2, a_3)$. This is canonical and so $h | \alpha$. This is Condition (2*ii*).

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$. Without loss of generality assume that $i = 0$, $j = 1$ and $k = 2$. Let $h' = \mathrm{hcf}(a_0, a_1)$. So $h' | d$. Hence the line P_0P_1 contains singularities of type $\frac{1}{h'}(a_2, a_3)$. As these are canonical $h = \mathrm{hcf}(h', a_2) = 1$. This is Condition (3).

□

In 1979, Reid produced the list of all families of codimension 1 weighted K3 surfaces; 95 in all (see [R1, section 4.5]). The full list follows along with their respective singularities.

Weighted K3 surface	Singularities	Weighted K3 surface	Singularities
X_4 in $\mathbf{P}(1, 1, 1, 1)$		X_5 in $\mathbf{P}(1, 1, 1, 2)$	A_1
X_6 in $\mathbf{P}(1, 1, 1, 3)$		X_6 in $\mathbf{P}(1, 1, 2, 2)$	$3{\times}A_1$
X_7 in $\mathbf{P}(1, 1, 2, 3)$	A_1, A_2	X_8 in $\mathbf{P}(1, 1, 2, 4)$	$2{\times}A_1$
X_8 in $\mathbf{P}(1, 2, 2, 3)$	$4{\times}A_1, A_2$	X_9 in $\mathbf{P}(1, 1, 3, 4)$	A_3
X_9 in $\mathbf{P}(1, 2, 3, 3)$	$A_1, 3{\times}A_2$	X_{10} in $\mathbf{P}(1, 1, 3, 5)$	A_2
X_{10} in $\mathbf{P}(1, 2, 2, 5)$	$5{\times}A_1$	X_{10} in $\mathbf{P}(1, 2, 3, 4)$	$2{\times}A_1, A_2, A_3$
X_{11} in $\mathbf{P}(1, 2, 3, 5)$	A_1, A_2, A_4	X_{12} in $\mathbf{P}(1, 1, 4, 6)$	A_1
X_{12} in $\mathbf{P}(1, 2, 3, 6)$	$2{\times}A_1, 2{\times}A_2$	X_{12} in $\mathbf{P}(1, 2, 4, 5)$	$3{\times}A_1, A_4$
X_{12} in $\mathbf{P}(1, 3, 4, 4)$	$3{\times}A_3$	X_{12} in $\mathbf{P}(2, 2, 3, 5)$	$6{\times}A_1, A_4$
X_{12} in $\mathbf{P}(2, 3, 3, 4)$	$3{\times}A_1, 4{\times}A_2$	X_{13} in $\mathbf{P}(1, 3, 4, 5)$	A_2, A_3, A_4
X_{14} in $\mathbf{P}(1, 2, 4, 7)$	$3{\times}A_1, A_3$	X_{14} in $\mathbf{P}(2, 2, 3, 7)$	$7{\times}A_1, A_2$
X_{14} in $\mathbf{P}(2, 3, 4, 5)$	$3{\times}A_1, A_2, A_3, A_4$	X_{15} in $\mathbf{P}(1, 2, 5, 7)$	A_1, A_6
X_{15} in $\mathbf{P}(1, 3, 4, 7)$	A_3, A_6	X_{15} in $\mathbf{P}(1, 3, 5, 6)$	$2{\times}A_2, A_5$
X_{15} in $\mathbf{P}(2, 3, 5, 5)$	$A_1, 3{\times}A_4$	X_{15} in $\mathbf{P}(3, 3, 4, 5)$	$5{\times}A_2, A_3$
X_{16} in $\mathbf{P}(1, 2, 5, 8)$	$2{\times}A_1, A_4$	X_{16} in $\mathbf{P}(1, 3, 4, 8)$	$A_2, 2{\times}A_3$
X_{16} in $\mathbf{P}(1, 4, 5, 6)$	A_1, A_4, A_5	X_{16} in $\mathbf{P}(2, 3, 4, 7)$	$4{\times}A_1, A_2, A_6$
X_{17} in $\mathbf{P}(2, 3, 5, 7)$	A_1, A_2, A_4, A_6	X_{18} in $\mathbf{P}(1, 2, 6, 9)$	$3{\times}A_1, A_2$
X_{18} in $\mathbf{P}(1, 3, 5, 9)$	$2{\times}A_2, A_4$	X_{18} in $\mathbf{P}(1, 4, 6, 7)$	A_3, A_1, A_6
X_{18} in $\mathbf{P}(2, 3, 4, 9)$	$4{\times}A_1, 2{\times}A_2, A_3$	X_{18} in $\mathbf{P}(2, 3, 5, 8)$	$2{\times}A_1, A_4, A_7$
X_{18} in $\mathbf{P}(3, 4, 5, 6)$	$3{\times}A_2, A_3, A_1, A_4$	X_{19} in $\mathbf{P}(3, 4, 5, 7)$	A_2, A_3, A_4, A_6
X_{20} in $\mathbf{P}(1, 4, 5, 10)$	$A_1, 2{\times}A_4$	X_{20} in $\mathbf{P}(2, 3, 5, 10)$	$2{\times}A_1, A_2, 2{\times}A_4$
X_{20} in $\mathbf{P}(2, 4, 5, 9)$	$5{\times}A_1, A_8$	X_{20} in $\mathbf{P}(2, 5, 6, 7)$	$3{\times}A_1, A_5, A_6$
X_{20} in $\mathbf{P}(3, 4, 5, 8)$	$A_2, 2{\times}A_3, A_7$	X_{21} in $\mathbf{P}(1, 3, 7, 10)$	A_9
X_{21} in $\mathbf{P}(1, 5, 7, 8)$	A_4, A_7	X_{21} in $\mathbf{P}(2, 3, 7, 9)$	$A_1, 2{\times}A_2, A_8$
X_{21} in $\mathbf{P}(3, 5, 6, 7)$	$3{\times}A_2, A_4, A_5$	X_{22} in $\mathbf{P}(1, 3, 7, 11)$	A_2, A_6
X_{22} in $\mathbf{P}(1, 4, 6, 11)$	A_3, A_1, A_5	X_{22} in $\mathbf{P}(2, 4, 5, 11)$	$5{\times}A_1, A_3, A_4$

Weighted K3 surface	Singularities	Weighted K3 surface	Singularities
X_{24} in $\mathbb{P}(1, 3, 8, 12)$	$2\times A_2, A_3$	X_{24} in $\mathbb{P}(1, 6, 8, 9)$	A_1, A_2, A_8
X_{24} in $\mathbb{P}(2, 3, 7, 12)$	$2\times A_1, 2\times A_2, A_6$	X_{24} in $\mathbb{P}(2, 3, 8, 11)$	$3\times A_1, A_{10}$
X_{24} in $\mathbb{P}(3, 4, 5, 12)$	$2\times A_2, 2\times A_3, A_4$	X_{24} in $\mathbb{P}(3, 4, 7, 10)$	A_1, A_6, A_9
X_{24} in $\mathbb{P}(3, 6, 7, 8)$	$4\times A_2, A_1, A_6$	X_{24} in $\mathbb{P}(4, 5, 6, 9)$	$2\times A_1, A_4, A_2, A_8$
X_{25} in $\mathbb{P}(4, 5, 7, 9)$	A_3, A_6, A_8	X_{26} in $\mathbb{P}(1, 5, 7, 13)$	A_4, A_6
X_{26} in $\mathbb{P}(2, 3, 8, 13)$	$3\times A_1, A_2, A_7$	X_{26} in $\mathbb{P}(2, 5, 6, 13)$	$4\times A_1, A_4, A_5$
X_{27} in $\mathbb{P}(2, 5, 9, 11)$	A_1, A_4, A_{10}	X_{27} in $\mathbb{P}(5, 6, 7, 9)$	A_4, A_5, A_2, A_6
X_{28} in $\mathbb{P}(1, 4, 9, 14)$	A_1, A_8	X_{28} in $\mathbb{P}(3, 4, 7, 14)$	$A_2, A_1, 2\times A_6$
X_{28} in $\mathbb{P}(4, 6, 7, 11)$	$2\times A_1, A_5, A_{10}$	X_{30} in $\mathbb{P}(1, 4, 10, 15)$	A_3, A_1, A_4
X_{30} in $\mathbb{P}(1, 6, 8, 15)$	A_1, A_2, A_7	X_{30} in $\mathbb{P}(2, 3, 10, 15)$	$3\times A_1, 2\times A_2, A_4$
X_{30} in $\mathbb{P}(2, 6, 7, 15)$	$5\times A_1, A_2, A_6$	X_{30} in $\mathbb{P}(3, 4, 10, 13)$	A_3, A_1, A_{12}
X_{30} in $\mathbb{P}(4, 5, 6, 15)$	$A_3, 2\times A_1, 2\times A_4, A_2$	X_{30} in $\mathbb{P}(5, 6, 8, 11)$	A_1, A_7, A_{10}
X_{32} in $\mathbb{P}(2, 5, 9, 16)$	$2\times A_1, A_4, A_8$	X_{32} in $\mathbb{P}(4, 5, 7, 16)$	$2\times A_3, A_4, A_6$
X_{33} in $\mathbb{P}(3, 5, 11, 14)$	A_4, A_{13}	X_{34} in $\mathbb{P}(3, 4, 10, 17)$	A_2, A_3, A_1, A_9
X_{34} in $\mathbb{P}(4, 6, 7, 17)$	$A_3, 2\times A_1, A_5, A_6$	X_{36} in $\mathbb{P}(1, 5, 12, 18)$	A_4, A_5
X_{36} in $\mathbb{P}(3, 4, 11, 18)$	$2\times A_2, A_1, A_{10}$	X_{36} in $\mathbb{P}(7, 8, 9, 12)$	A_6, A_7, A_3, A_2
X_{38} in $\mathbb{P}(3, 5, 11, 19)$	A_2, A_4, A_{10}	X_{38} in $\mathbb{P}(5, 6, 8, 19)$	A_4, A_5, A_1, A_7
X_{40} in $\mathbb{P}(5, 7, 8, 20)$	$2\times A_4, A_6, A_3$	X_{42} in $\mathbb{P}(1, 6, 14, 21)$	A_1, A_2, A_6
X_{42} in $\mathbb{P}(2, 5, 14, 21)$	$3\times A_1, A_4, A_6$	X_{42} in $\mathbb{P}(3, 4, 14, 21)$	$2\times A_2, A_3, A_1, A_6$
X_{44} in $\mathbb{P}(4, 5, 13, 22)$	A_1, A_4, A_{12}	X_{48} in $\mathbb{P}(3, 5, 16, 24)$	$2\times A_2, A_4, A_7$
X_{50} in $\mathbb{P}(7, 8, 10, 25)$	A_6, A_7, A_1, A_4	X_{54} in $\mathbb{P}(4, 5, 18, 27)$	A_3, A_1, A_4, A_8
X_{66} in $\mathbb{P}(5, 6, 22, 33)$	A_4, A_1, A_2, A_{10}		

However there are not so many dimension 2 weighted hypersurfaces with $\omega_S = \mathcal{O}_S(\pm 1)$:

III.5.3 Theorem: *There are exactly 3 families of dimension 2 weighted hypersurfaces with at worst canonical singularities and $\omega_S = \mathcal{O}_S(1)$, and exactly 3 families with $\omega_S = \mathcal{O}_S(-1)$.*

$\alpha = 1$	$\alpha = -1$
S_5 *in* $\mathbb{P}(1, 1, 1, 1)$	S_3 *in* $\mathbb{P}(1, 1, 1, 1)$
S_6 *in* $\mathbb{P}(1, 1, 1, 2)$	S_4 *in* $\mathbb{P}(1, 1, 1, 2)$
S_8 *in* $\mathbb{P}(1, 1, 1, 4)$	S_6 *in* $\mathbb{P}(1, 1, 2, 3)$

III.5.4 Note. These families are all nonsingular.

Proof. Condition ($2ii$) of Theorem III.5.1 is very strong when $\alpha = \pm 1$ and forces the $\{a_i\}$ to be pairwise coprime. Similarly condition ($1iv$) forces $a_i | d$ for each i. So $a_0 a_1 a_2 a_3 | d$ and $d = a_0 + \ldots + a_3 + \alpha$. Order $a_3 \geq a_2 \geq a_1 \geq a_0 \geq 1$ and let $d = \lambda a_3$. Thus $a_0 a_1 a_2 | \lambda$ and $(\lambda - 1) a_3 = a_0 + \ldots + a_2 + \alpha$.

Suppose $\alpha = 1$. Then $2a_3 \leq \lambda a_3 = a_0 + \ldots + a_3 + 1 \leq 5a_3$. So $2 \leq \lambda \leq 5$. Running through the possible values of λ:

(i) $\lambda = 5$.

So $a_0 a_1 a_2 | 5$. If $a_2 = 1$ then $a_4 = 1$ (i.e. S_5 in $\mathbb{P}(1, 1, 1, 1)$). If $a_2 = 5$ then $a_3 = 2$, a contradiction.

(ii) $\lambda = 4$.

So $a_0 a_1 a_2 | 4$. If $a_2 = 1$ then $a_4 = \frac{4}{3}$, a contradiction. If $a_2 = 2$ then $a_4 = \frac{5}{3}$, a contradiction. If $a_2 = 4$ then $a_4 = \frac{7}{3}$, a contradiction.

(iii) $\lambda = 3$.

So $a_0\, a_1\, a_2 | 3$. If $a_2 = 1$ then $a_4 = 2$ (i.e. S_6 in $\mathbf{P}(1, 1, 1, 2)$). If $a_2 = 3$ then $a_4 = 3$, a contradiction.

(iv) $\lambda = 2$.

So $a_0\, a_1\, a_2 | 2$. If $a_2 = 1$ then $a_4 = 4$ (i.e. S_8 in $\mathbf{P}(1, 1, 1, 4)$). If $a_2 = 2$ then $a_4 = \frac{5}{2}$, a contradiction.

So there are exactly 3 families.

Suppose that $\alpha = -1$. Then $2a_3 \le \lambda a_3 = a_0 + \ldots + a_3 - 1 \le 6a_3$. Thus $2 \le \lambda \le 6$. As above this gives rise to the following families: S_3 in $\mathbf{P}(1, 1, 1, 1)$ in the case $\lambda = 3$, S_4 in $\mathbf{P}(1, 1, 1, 2)$ and S_6 in $\mathbf{P}(1, 1, 2, 3)$ in the case $\lambda = 2$.

□

Consider the case of codimension 2 complete intersections.

III.5.5 Theorem: *Suppose $S = S_{d_1, d_2}$ in $\mathbf{P} = \mathbf{P}(a_0, \ldots, a_4)$ is quasismooth and not the intersection of a linear cone with another hypersurface. Let $\alpha = \sum d_\lambda - \sum a_i$. S is well-formed and has at worst canonical singularities if and only if the following hold:*

(1) for all i, if $a_i \nmid d_1$ and $a_i \nmid d_2$ then $a_i | \alpha$.

(2) for all distinct i and j, with $h = \mathrm{hcf}(a_i, a_j)$, one of the following occurs:

(a) $h | d_1$ and $h | d_2$,

(b) $h | d_1$, $h \nmid d_2$ and $h | \alpha$, or

(c) $h \nmid d_1$, $h | d_2$ and $h | \alpha$.

(3) for all distinct i, j and k, with $h = \mathrm{hcf}(a_i, a_j, a_k)$, $h | d_1$, $h | d_2$ and $h | \alpha$.

(4) for all distinct i, j, k and l, $h = \mathrm{hcf}(a_i, a_j, a_k, a_l) = 1$

III.5.6 Note. Since the hypersurface is well-formed we have that $\omega_S = \mathcal{O}_S(\alpha)$.

Proof. Let f_1 and f_2 be sufficiently general homogeneous polynomials of degrees d_1 and d_2 respectively, in the variables $x_0, \ldots, x_4$ with respect to the weights $a_0, \ldots, a_4$. Define $S : (f_1 = 0, f_2 = 0) \subset \mathbf{P}$.

Since S is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are all cyclic quotient singularities occurring on the fundamental simplex Δ.

Assume Conditions (1), ..., (4) hold. By Conditions (2), (3) and (4) S is well-formed. By Condition (4) only the vertices, edges and faces of Δ need be considered.

Suppose $P_i \in S$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d_1$ and $a_0 \nmid d_2$. As S is quasismooth (and using $I = \{0\}$ in Theorem III.3.7) there exist monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \ne e_2$. By renumbering we can write $e_1 = 1$ and $e_2 = 2$. So f_1 and f_2 are of the form:

$$f_1 = x_0^n x_1 + \ldots$$

$$f_2 = x_0^m x_2 + \ldots$$

Thus $\partial f_1 / \partial x_1$ and $\partial f_2 / \partial x_2$ are nonzero at P_0. By the Inverse Function Theorem, x_3 and x_4

are local coordinates around P_0. Hence $P_0 \in S$ is of type $\frac{1}{a_0}(a_3, a_4)$. As $d_1 + d_2 = a_0 + \ldots + a_4 + \alpha$ and $a_0 | \alpha$ then $a_0 | a_3 + a_4$. Let $h = \mathrm{hcf}(a_0, a_3)$. So $h | a_4$ and, by Condition (3), $h | d_1$. Since $\deg(x_0^n x_1) = d_1$, $h | a_1$ and so, by Condition (4), $h = 1$. Thus $P_0 \in S$ is canonical.

Consider the edge $P_i P_j$. By renumbering we can assume that $i = 0$ and $j = 1$. Let $h = \mathrm{hcf}(a_0, a_1)$. Notice that $P_0 P_1 \subset X_{d_\lambda}$ if and only if $h \nmid d_\lambda$ for $\lambda = 0, 1$. By Condition (2), $h | d_\lambda$ for some λ. Without loss of generality assume that $h | d_1$. There are 2 cases:

(a) $h | d_2$.

$P_0 P_1 \cap (f_\lambda = 0)$ is a finite set of points for $\lambda = 0, 1$. Thus $P_0 P_1 \cap S = \varnothing$.

(b) $h \nmid d_2$.

In this case no monomial of the form $x_0^n x_1^m$ of degree d_2 exists (or else $h | d_2$). From Theorem III.3.7 (with $I = \{0, 1\}$) there exists a monomial $x_0^n x_1^m x_e$ of degree d_2, where $e \neq 0, 1$. By renumbering we can assume that $e = 2$. Thus f_2 is of the form:

$$f_2 = x_0^n x_1^m x_2 + \ldots$$

and $\partial f_2 / \partial x_2$ is nonzero on $P_0 P_1 \cap S$. By the Inverse Function Theorem, x_3 and x_4 are local coordinates around each point of $P_0 P_1 \cap S$ and so each is of type $\frac{1}{h}(a_3, a_4)$. Condition ($2b$) gives $h | \alpha$ and so $h | a_3 + a_4$. Let $h' = \mathrm{hcf}(h, a_3)$. So $h | a_4$ and thus by Condition (4) $h = 1$. Thus these points are canonical.

Therefore S has at worst canonical points along $P_0 P_1$.

Consider the face $P_i P_j P_k$. As before assume $i = 0$, $j = 1$ and $k = 2$. By Condition (3) $h = \mathrm{hcf}(a_0, a_1, a_2) | d_1$ and $h | d_2$. So $P_0 P_1 P_2$ intersects S transversally. Each point in the intersection is of type $\frac{1}{h}(a_3, a_4)$. As $h | \alpha$, $h | a_3 + a_4$. By Condition (4) $\mathrm{hcf}(h, a_3) = 1$. Thus these points are canonical.

Therefore Conditions (1), ..., (4) are sufficient.

Conversely assume that S is well-formed and has at worst canonical singularities. Suppose $a_i \nmid d_1$ and $a_i \nmid d_2$. By renumbering assume $i = 0$. Thus $P_0 \in S$. Since S is quasismooth there exist 2 monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. Without loss of generality we can assume that $e_1 = 1$ and $e_2 = 2$. As before we find that $P_0 \in S$ is of type $\frac{1}{a_0}(a_3, a_4)$. As this is canonical $a_0 | a_3 + a_4$ and so $a_0 | \alpha$. This is Condition (1).

Suppose $h = \mathrm{hcf}(a_i, a_j)$ for distinct i and j. As usual we can renumber such that $i = 0$ and $j = 1$. As S is well-formed then $h | d_\lambda$ for some λ. Suppose $h | d_1$. If $h | d_2$ then this is Condition ($2a$). So assume that $h \nmid d_2$. As above each point of $P_0 P_1 \cap S$ is isolated and of type $\frac{1}{h}(a_3, a_4)$. Thus $h | a_3 + a_4$ and so $h | \alpha$. This is Condition ($2b$). Likewise for the case when $h | d_2$ but $h \nmid d_1$. This gives Condition ($2c$).

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$ for distinct i, j and k. Renumber such that $i = 0$, $j = 1$ and $k = 2$. As S is well-formed then $h | d_1$ and $h | d_2$. Thus $P_0 P_1 P_2 \cap S$ is a finite number of points, all of type $\frac{1}{h}(a_3, a_4)$. As these are canonical $h | a_3 + a_4$ and so $h | \alpha$. This is Condition (3). Also $\mathrm{hcf}(h, a_3) = \mathrm{hcf}(h, a_4) = 1$, which is Condition (4).

So these conditions are necessary.

□

III.5.7 Codimension 2 Weighted K3 Surfaces.

There are 84 families of codimension 2 quasismooth, well-formed K3 surfaces with only canonical singularities and $\sum a_i \leq 100$. These were found by means of a computer search program similar to that used to search for 3-fold complete intersections (see section III.10.7).

Weighted K3 surfaces	Singularities
$X_{2,3}$ in $\mathbb{P}(1,1,1,1,1)$	
$X_{3,3}$ in $\mathbb{P}(1,1,1,1,2)$	A_1
$X_{3,4}$ in $\mathbb{P}(1,1,1,2,2)$	$2\times A_1$
$X_{4,4}$ in $\mathbb{P}(1,1,1,2,3)$	A_2
$X_{4,4}$ in $\mathbb{P}(1,1,2,2,2)$	$4\times A_1$
$X_{4,5}$ in $\mathbb{P}(1,1,2,2,3)$	$2\times A_1, A_2$
$X_{4,6}$ in $\mathbb{P}(1,1,2,3,3)$	$2\times A_2$
$X_{4,6}$ in $\mathbb{P}(1,2,2,2,3)$	$6\times A_1$
$X_{5,6}$ in $\mathbb{P}(1,1,2,3,4)$	A_1, A_3
$X_{5,6}$ in $\mathbb{P}(1,2,2,3,3)$	$3\times A_1, 2\times A_2$
$X_{6,6}$ in $\mathbb{P}(1,1,2,3,5)$	A_4
$X_{6,6}$ in $\mathbb{P}(1,2,2,3,4)$	$4\times A_1, A_3$
$X_{6,6}$ in $\mathbb{P}(1,2,3,3,3)$	$4\times A_2$
$X_{6,6}$ in $\mathbb{P}(2,2,2,3,3)$	$9\times A_1$
$X_{6,7}$ in $\mathbb{P}(1,2,2,3,5)$	$3\times A_1, A_4$
$X_{6,7}$ in $\mathbb{P}(1,2,3,3,4)$	$A_1, 2\times A_2, A_3$
$X_{6,8}$ in $\mathbb{P}(1,1,3,4,5)$	A_4
$X_{6,8}$ in $\mathbb{P}(1,2,3,3,5)$	$2\times A_2, A_4$
$X_{6,8}$ in $\mathbb{P}(1,2,3,4,4)$	$2\times A_1, 2\times A_3$
$X_{6,8}$ in $\mathbb{P}(2,2,3,3,4)$	$6\times A_1, 2\times A_2$
$X_{6,9}$ in $\mathbb{P}(1,2,3,4,5)$	A_1, A_3, A_4
$X_{7,8}$ in $\mathbb{P}(1,2,3,4,5)$	$2\times A_1, A_2, A_4$
$X_{6,10}$ in $\mathbb{P}(1,2,3,5,5)$	$2\times A_4$
$X_{6,10}$ in $\mathbb{P}(2,2,3,4,5)$	$7\times A_1, A_3$
$X_{8,9}$ in $\mathbb{P}(1,2,3,4,7)$	$2\times A_1, A_6$
$X_{8,9}$ in $\mathbb{P}(1,3,4,4,5)$	$2\times A_3, A_4$
$X_{8,9}$ in $\mathbb{P}(2,3,3,4,5)$	$2\times A_1, 3\times A_2, A_4$
$X_{8,10}$ in $\mathbb{P}(1,2,3,5,7)$	A_2, A_6
$X_{8,10}$ in $\mathbb{P}(1,2,4,5,6)$	$3\times A_1, A_5$
$X_{8,10}$ in $\mathbb{P}(1,3,4,5,5)$	$A_2, 2\times A_4$
$X_{8,10}$ in $\mathbb{P}(2,3,4,4,5)$	$4\times A_1, A_2, 2\times A_3$
$X_{9,10}$ in $\mathbb{P}(1,2,3,5,8)$	A_1, A_7
$X_{9,10}$ in $\mathbb{P}(1,3,4,5,6)$	A_2, A_3, A_5
$X_{9,10}$ in $\mathbb{P}(2,2,3,5,7)$	$5\times A_1, A_6$
$X_{9,10}$ in $\mathbb{P}(2,3,4,5,5)$	$2\times A_1, A_3, 2\times A_4$
$X_{8,12}$ in $\mathbb{P}(1,3,4,5,7)$	A_4, A_6
$X_{8,12}$ in $\mathbb{P}(2,3,4,5,6)$	$4\times A_1, 2\times A_2, A_4$

Weighted K3 surfaces	Singularities
$X_{9,12}$ in $\mathbb{P}(2,3,4,5,7)$	$3\times A_1, A_4, A_6$
$X_{10,11}$ in $\mathbb{P}(2,3,4,5,7)$	$2\times A_1, A_2, A_3, A_6$
$X_{10,12}$ in $\mathbb{P}(1,3,4,5,9)$	A_2, A_8
$X_{10,12}$ in $\mathbb{P}(1,3,5,6,7)$	$2\times A_2, A_6$
$X_{10,12}$ in $\mathbb{P}(1,4,5,6,6)$	$A_1, 2\times A_5$
$X_{10,12}$ in $\mathbb{P}(2,3,4,5,8)$	$3\times A_1, A_3, A_7$
$X_{10,12}$ in $\mathbb{P}(2,3,5,5,7)$	$2\times A_4, A_6$
$X_{10,12}$ in $\mathbb{P}(2,4,5,5,6)$	$5\times A_1, 2\times A_4$
$X_{10,12}$ in $\mathbb{P}(3,3,4,5,7)$	$4\times A_2, A_6$
$X_{10,12}$ in $\mathbb{P}(3,4,4,5,6)$	$2\times A_2, 3\times A_3, A_1$
$X_{11,12}$ in $\mathbb{P}(1,4,5,6,7)$	A_1, A_4, A_6
$X_{10,14}$ in $\mathbb{P}(1,2,5,7,9)$	A_8
$X_{10,14}$ in $\mathbb{P}(2,3,5,7,7)$	$A_2, 2\times A_6$
$X_{10,14}$ in $\mathbb{P}(2,4,5,6,7)$	$5\times A_1, A_3, A_5$
$X_{10,15}$ in $\mathbb{P}(2,3,5,7,8)$	A_1, A_6, A_7
$X_{12,13}$ in $\mathbb{P}(3,4,5,6,7)$	$2\times A_2, A_1, A_4, A_6$
$X_{12,14}$ in $\mathbb{P}(1,3,4,7,11)$	A_{10}
$X_{12,14}$ in $\mathbb{P}(1,4,6,7,8)$	A_1, A_3, A_7
$X_{12,14}$ in $\mathbb{P}(2,3,4,7,10)$	$4\times A_1, A_9$
$X_{12,14}$ in $\mathbb{P}(2,3,5,7,9)$	A_2, A_4, A_8
$X_{12,14}$ in $\mathbb{P}(3,4,5,7,7)$	$A_4, 2\times A_6$
$X_{12,14}$ in $\mathbb{P}(4,4,5,6,7)$	$3\times A_3, 2\times A_1, A_4$
$X_{12,15}$ in $\mathbb{P}(1,4,5,6,11)$	A_1, A_{10}
$X_{12,15}$ in $\mathbb{P}(3,4,5,6,9)$	$3\times A_2, A_1, A_8$
$X_{12,15}$ in $\mathbb{P}(3,4,5,7,8)$	A_3, A_6, A_7
$X_{12,16}$ in $\mathbb{P}(2,5,6,7,8)$	$4\times A_1, A_4, A_6$
$X_{14,15}$ in $\mathbb{P}(2,3,5,7,12)$	A_1, A_2, A_{11}
$X_{14,15}$ in $\mathbb{P}(2,5,6,7,9)$	$2\times A_1, A_5, A_8$
$X_{14,15}$ in $\mathbb{P}(3,4,5,7,10)$	A_3, A_4, A_9
$X_{14,15}$ in $\mathbb{P}(3,5,6,7,8)$	$2\times A_2, A_5, A_7$
$X_{14,16}$ in $\mathbb{P}(1,5,7,8,9)$	A_4, A_8
$X_{14,16}$ in $\mathbb{P}(3,4,5,7,11)$	A_2, A_4, A_{10}
$X_{14,16}$ in $\mathbb{P}(4,5,6,7,8)$	$A_1, 2\times A_3, A_4, A_5$
$X_{15,16}$ in $\mathbb{P}(2,3,5,8,13)$	$2\times A_1, A_{12}$
$X_{15,16}$ in $\mathbb{P}(3,4,5,8,11)$	$2\times A_3, A_{10}$
$X_{14,18}$ in $\mathbb{P}(2,3,7,9,11)$	$2\times A_2, A_{10}$
$X_{14,18}$ in $\mathbb{P}(2,6,7,8,9)$	$5\times A_1, A_2, A_7$
$X_{12,20}$ in $\mathbb{P}(4,5,6,7,10)$	$2\times A_1, 2\times A_4, A_6$
$X_{16,18}$ in $\mathbb{P}(1,6,8,9,10)$	A_1, A_2, A_9
$X_{16,18}$ in $\mathbb{P}(4,6,7,8,9)$	$2\times A_1, 2\times A_3, A_2, A_6$
$X_{18,20}$ in $\mathbb{P}(4,5,6,9,14)$	$2\times A_1, A_2, A_{13}$
$X_{18,20}$ in $\mathbb{P}(4,5,7,9,13)$	A_6, A_{12}
$X_{18,20}$ in $\mathbb{P}(5,6,7,9,11)$	A_2, A_6, A_{10}
$X_{18,22}$ in $\mathbb{P}(2,5,9,11,13)$	A_4, A_{12}
$X_{20,21}$ in $\mathbb{P}(3,4,7,10,17)$	A_1, A_{16}

Weighted K3 surfaces	Singularities
$X_{18,30}$ in $\mathbb{P}(6, 8, 9, 10, 15)$	$2\times A_1, 2\times A_2, A_7, A_4$
$X_{24,30}$ in $\mathbb{P}(8, 9, 10, 12, 15)$	A_1, A_3, A_8, A_2, A_4

III.6 Weighted 3-fold complete intersections.

This section gives the corresponding conditions and lists for 3-folds.

III.6.1 Theorem: *Let X_d be a general hypersurface in $\mathbf{P} = \mathbf{P}(a_0, \ldots, a_4)$ and let $\alpha = d - \sum a_i$. X_d is quasismooth, well-formed with only isolated terminal quotient singularities and is not a linear cone if and only if all the following hold:*

(1) For all i,

(i) $d > a_i$.

(ii) there exists a monomial $x_i^m x_e$ of degree d (i.e. there exists e such that $a_i | d - a_e$).

(iii) if $a_i \nmid d$, there exists an $m \neq i$, e such that $a_i | a_m + \alpha$.

(2) For all distinct i, j with $h = \mathrm{hcf}(a_i, a_j)$

(i) then $h | d$.

(ii) there exists an $m \neq i, j$ such that $h | a_m + \alpha$.

(iii) one of the following holds:

either

there exists a monomial $x_i^m x_j^n$ of degree d,

or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.

(iv) there exists a monomial of degree d which does not involve x_i or x_j.

(3) For all distinct i, j, k, $\mathrm{hcf}(a_i, a_j, a_k) = 1$.

III.6.2 Note. Since the hypersurface is well-formed then $\omega_X = \mathcal{O}_X(\alpha)$.

Proof. Let f be a general homogeneous polynomial of degree d in variables $x_0, \ldots, x_3$; define $X_d : (f = 0) \subset \mathbf{P}$.

X_d is quasismooth, well-formed and not a linear cone if and only if Conditions (1*i*), (1*ii*), (2*i*), (2*iii*), (2*iv*) and (3) hold (see Corollary III.3.6). By calculating the types of the singularities on X_d we can show that Conditions ((1*iii*), (2*i*), (2*ii*) and (3) are equivalent to these singularities being terminal; the combinatorial conditions for which are found in Lemma III.3.11.

Suppose furthermore that Conditions (1*iii*), (2*i*), (2*ii*) and (3) hold. As X_d is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities on the fundamental simplex $\Delta \subset \mathbf{P}$. By Condition (3) only vertices and edges need be checked.

Consider $P_i \in X_d$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d$. Condition (1*ii*) gives that there exists an $e \neq 0$ such that $a_0 | d - a_e$. Without loss of generality we can assume that $e = 1$. So f is of the form $f = x_0^n x_1 + \ldots$. Thus $\partial f / \partial x_1$ is nonzero at P_0. By the Inverse Function Theorem x_2, x_3 and x_4 are local coordinates around P_0. So $P_0 \in X_d$ is of type $\frac{1}{a_0}(a_2, a_3, a_4)$. However $d = a_0 + \ldots + a_4 + \alpha$ and so $a_0 | a_2 + a_4 + \alpha$. By Condition (1*iv*), $a_0 | \alpha + a_m$ for some $m = 2, 3, 4$. Without loss of generality assume $m = 2$. By Condition (1*iv*), $a_0 | a_3 + a_4$. Let $h = \text{hcf}(a_0, a_3)$. So $h | a_3$ and hence, by Condition (3), $h = 1$. Therefore $P_0 \in X_d$ is a terminal singularity.

Consider the edge P_iP_j. Again by renumbering assume that $i = 0$ and $j = 1$. f restricted to P_0P_1 is:

$$f = \sum x_0^n x_1^m$$

where the sum is taken over the set $\{(n, m) : na_0 + ma_1 = d\}$. If $a_0 \nmid d$ then $a_0 | d - a_e$ for some $e \neq 0$. If $e \neq 1$ then $h = \text{hcf}(a_0, a_1) | a_e$ and by Condition (4) $h = 1$. Then P_0P_1 is nonsingular. So assume that either $a_0 | d$ or $a_0 | d - a_1$. Hence f is not identically zero on P_0P_1, and so $X_d \cap P_0P_1$ is finite. Each point in this intersection is of type $\frac{1}{h}(a_2, a_3, a_4)$. By Condition (2*ii*) $h | \alpha + a_m$ for some $m = 2, 3, 4$. By renumbering assume $m = 2$. Since $d = a_0 + \ldots + a_4 + \alpha$, then $h | a_3 + a_4$. Also $\text{hcf}(h, a_3) = 1$. Thus each point is terminal.

Therefore X_d in $\mathbf{P}$ has at worst terminal singularities.

Conversely assume that X_d is quasismooth, not a linear cone and has at worst only terminal singularities. Suppose $a_i \nmid d$. By renumbering we can assume that $i = 0$. So $P_0 \in X_d$ and $a_0 | d - a_e$ for some e. Without loss of generality assume that $e = 1$. As above the singularity at $P_0 \in X_d$ is of type $\frac{1}{a_0}(a_2, a_3, a_4)$. Since this is terminal we have, after renumbering, $a_0 | a_2 + a_3$ and so $a_0 | \alpha + a_m$ for some m. This is Condition (1*iv*).

Suppose $h = \text{hcf}(a_i, a_j)$. By renumbering assume that $i = 0$ and $j = 1$. As X_d is well-formed then $h | d$, which is Condition (2*i*). So $P_0P_1 \cap X_d$ is a finite intersection, where each point is of type $\frac{1}{h}(a_2, a_3, a_4)$. This is terminal and so $h | \alpha + a_m$ for $m = 2, 3, 4$. This is Condition (2*ii*).

Suppose $h = \text{hcf}(a_i, a_j, a_k)$. Without loss of generality assume that $i = 0$, $j = 1$ and $k = 2$. Let $h' = \text{hcf}(a_0, a_1)$. So $h' | d$. Hence the line P_0P_1 contains singularities of type $\frac{1}{h'}(a_2, a_3, a_4)$. As these are terminal $h = \text{hcf}(h', a_2) = 1$. This is Condition (3).

$\square$

III.6.3 Theorem: *There are exactly 4 families of quasismooth, well-formed 3-fold weighted hypersurfaces with only terminal isolated quotient singularities and $\omega_X \cong \mathcal{O}_X$:*

X_5 *in* $\mathbf{P}(1, 1, 1, 1, 1)$
X_6 *in* $\mathbf{P}(1, 1, 1, 1, 2)$
X_8 *in* $\mathbf{P}(1, 1, 1, 1, 4)$
X_{10} *in* $\mathbf{P}(1, 1, 1, 2, 5)$

Notice that the above are all non-singular.

Proof. As $K_X \cong O_X$ then $\alpha = 0$. Suppose $h = \mathrm{hcf}(a_i, a_j) \neq 1$ for distinct i, j. By Theorem III.6.1 (2*ii*) there exists an $m \neq i, j$ such that $h|a_m + \alpha$. However $\alpha = 0$ and so $h|a_m$. By (3) $h = 1$, a contradiction. Hence a_i and a_j are coprime for distinct i, j.

Suppose that $a_i \nmid d$. Then there exists an $m \neq i, e_i$ such that $a_i|a_m + \alpha$. Thus $a_i = \mathrm{hcf}(a_i, a_m) = 1$, contradicting $a_i \nmid d$. Thus $a_i|d$ for all i.

Order the $\{a_i\}$ such that $a_4 \geq ... \geq a_0$. So $5a_4 \geq d \geq 2a_4$. Let $d = ka_4$. As the $\{a_i\}$ are pairwise coprime then $a_0 a_1 a_2 a_3 a_4 | d$. So $a_0 a_1 a_2 a_3 | k$. Also $a_0 + ... + a_3 = (k-1)a_4$. There are four cases:

(i) $k = 5$.
Either $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$ giving $a_4 = 1$ (i.e. X_5 in $\mathbb{P}(1, 1, 1, 1, 1)$) or $(a_0, a_1, a_2, a_3) = (1, 1, 1, 5)$ giving $a_4 = 2 < a_3$.

(ii) $k = 4$.
Either $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$ giving $3 \nmid 4$, or $(a_0, a_1, a_2, a_3) = (1, 1, 1, 2)$ giving $3 \nmid 5$, or $(a_0, a_1, a_2, a_3) = (1, 1, 1, 4)$ giving $3 \nmid 7$, all contradictions.

(iii) $k = 3$.
Either $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$ giving $a_4 = 2$ (i.e. X_6 in $\mathbb{P}(1, 1, 1, 1, 2)$), or $(a_0, a_1, a_2, a_3) = (1, 1, 1, 3)$ giving $a_4 = 3$, contradicting the coprime condition.

(iv) $k = 2$.
Either $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$ giving $a_4 = 4$ (i.e. X_8 in $\mathbb{P}(1, 1, 1, 1, 4)$), or $(a_0, a_1, a_2, a_3) = (1, 1, 1, 2)$ giving $a_4 = 5$ (i.e. X_{10} in $\mathbb{P}(1, 1, 1, 2, 5)$).

□

Consider the case of codimension 2 complete intersections.

III.6.4 Theorem: *Suppose* $X = X_{d_1, d_2}$ *in* $\mathbb{P} = \mathbb{P}(a_0, ..., a_5)$ *is quasismooth and not the intersection of a linear cone with another hypersurface. Let* $\alpha = \sum d_\lambda - \sum a_i$. *X is well-formed and has at worst terminal singularities if and only if the following hold:*

(1) for all i, *if* $a_i \nmid d_1$ *and* $a_i \nmid d_2$ *then there exists* e_1, e_2 *and* m *such that* $a_i | d_1 - a_{e_1}$, $a_i | d_2 - a_{e_2}$ *and* $a_i | \alpha + a_m$, *where* $\{i, e_1, e_2, m\}$ *are distinct.*

(2) for all distinct i *and* j, *with* $h = \mathrm{hcf}(a_i, a_j)$, *one of the following occurs:*

(a) $h|d_1$ *and* $h|d_2$,

(b) $h|d_1$, $h \nmid d_2$ *and* $h|\alpha + a_m$ *for some* $m \neq i, j$, *or*

(c) $h \nmid d_1$, $h|d_2$ *and* $h|\alpha + a_m$ *for some* $m \neq i, j$.

(3) for all distinct i, j *and* k, *with* $h = \mathrm{hcf}(a_i, a_j, a_k)$, $h|d_1$, $h|d_2$ *and* $h|\alpha + a_m$ *for some* $m \neq i, j, k$.

(4) for all distinct i, j, k *and* l, $h = \mathrm{hcf}(a_i, a_j, a_k, a_l) = 1$.

III.6.5 Note. Since the hypersurface is well-formed we have $\omega_X = O_X(\alpha)$.

Proof. Let f_1 and f_2 be sufficiently general homogeneous polynomials of degrees d_1 and d_2 respectively, in the variables $x_0, \ldots, x_4$ with respect to the weights $a_0, \ldots, a_4$. Define $X : (f_1 = 0, f_2 = 0) \subset \mathbf{P}$.

Since X is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are all cyclic quotient singularities occurring on the fundamental simplex Δ.

Assume Conditions (1), ..., (4) hold. By Condition (4) only the vertices, edges and faces of Δ need be considered.

Suppose $P_i \in X$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d_1$ and $a_0 \nmid d_2$. By Condition (1), there exist monomials $x_0^{n_1}x_{e_1}$ and $x_0^{n_2}x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. Note that this is really quasismoothness. By renumbering we can write $e_1 = 1$ and $e_2 = 2$. So f_1 and f_2 are of the form:

$$f_1 = x_0^{n_1}x_1 + \ldots$$

$$f_2 = x_0^{n_2}x_2 + \ldots$$

Thus $\partial f_1/\partial x_1$ and $\partial f_2/\partial x_2$ are nonzero at P_0. By the Inverse Function Theorem, x_3, x_4 and x_5 are local coordinates. Hence $P_0 \in X$ is of type $\frac{1}{a_0}(a_3, a_4, a_5)$. By Condition (1) $a_0 | \alpha + a_m$ for some $m \neq 0, 1, 2$. Without loss of generality assume $m = 3$. As $d_1 + d_2 = a_0 + \ldots + a_5 + \alpha$ then $a_0 | a_4 + a_5$. Let $h = \mathrm{hcf}(a_0, a_4)$. So $h | a_5$ and, by Condition (3), $h | d_1$. Since $\deg x_0^n x_1 = d_1$, $h | a_1$ and so, by Condition (4), $h = 1$. Thus $P_0 \in X$ is terminal.

Consider the edge P_iP_j. By renumbering we can assume that $i = 0$ and $j = 1$. Let $h = \mathrm{hcf}(a_0, a_1)$. Notice that $P_0P_1 \subset X_{d_\lambda}$ if and only if $h \nmid d_\lambda$ for $\lambda = 0, 1$. By Condition (2), $h | d_\lambda$ for some λ. Without loss of generality assume that $h | d_1$. There are 2 cases:

(a) $h | d_2$.

$P_0P_1 \cap (f_\lambda = 0)$ is a finite set of points for $\lambda = 0, 1$. Thus $P_0P_1 \cap X = \varnothing$.

(b) $h \nmid d_2$.

In this case no monomial of the form $x_0^{n_0}x_1^{n_1}$ of degree d_2 exists (or else $h | d_2$). From Theorem III.3.7 (with $I = \{0, 1\}$) there exists a monomial $x_0^{n_0}x_1^{n_1}x_e$ of degree d_2, where $e \neq 0, 1$. By renumbering we can assume that $e = 2$. Thus f_2 is of the form:

$$f_2 = x_0^{n_0}x_1^{n_1}x_2 + \ldots$$

and $\partial f_2/\partial x_2$ is nonzero on $P_0P_1 \cap X$. By the Inverse Function Theorem, x_3, x_4 and x_5 are local coordinates and so each point of $P_0P_1 \cap X$ is of type $\frac{1}{h}(a_3, a_4, a_5)$. Condition (2*b*) gives $h | \alpha + a_m$ for some $m \neq 0, 1, 2$. Assume that $m = 3$. and so $h | a_4 + a_5$. Let $h' = \mathrm{hcf}(h, a_4)$. So $h | a_4$ and thus by Condition (4) $h = 1$. Thus these points are terminal.

Therefore X has at worst terminal points along P_0P_1.

Consider the face $P_iP_jP_k$. As before assume $i = 0$, $j = 1$ and $k = 2$. By Condition (3) $h = \mathrm{hcf}(a_0, a_1, a_2) | d_1$ and $h | d_2$. So $P_0P_1P_2$ intersects X transversally. Each point in the intersection is of type $\frac{1}{h}(a_3, a_4, a_5)$. As $h | \alpha + a_m$ for some $m \neq 0, 1, 2$, after renumbering, $h | a_3 + a_4$. By Condition (4) $\mathrm{hcf}(h, a_3) = 1$. Thus these points are terminal.

Therefore Condition (1), ..., (4) are sufficient.

Conversely assume that X has at worst terminal singularities. Suppose $a_i \nmid d_1$ and $a_i \nmid d_2$. By renumbering assume $i = 0$. Thus $P_0 \in X$. Since X is quasismooth there exist 2 monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. This gives the first part of Condition (1). Without loss of generality we can assume that $e_1 = 1$ and $e_2 = 2$. As before we find that $P_0 \in X$ is of type $\frac{1}{a_0}(a_3, a_4, a_5)$. As this is terminal, after renumbering, $a_0 | a_3 + a_4$ and so $a_0 | \alpha + a_5$. This is Condition (1).

Suppose $h = \mathrm{hcf}(a_i, a_j)$ for distinct i and j. As usual we can renumber such that $i = 0$ and $j = 1$. As X is well-formed then $h | d_\lambda$ for some λ. Suppose $h | d_1$. If $h | d_2$ then this is Condition (2a). So assume that $h \nmid d_2$. As above each point of $P_0P_1 \cap X$ is isolated and of type $\frac{1}{h}(a_3, a_4, a_5)$. After renumbering, $h | a_3 + a_4$ and so $h | \alpha + a_5$. This is Condition (2b). Likewise for the case when $h | d_2$ but $h \nmid d_1$. This gives Condition (2c).

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$ for distinct i, j and k. Renumber such that $i = 0$, $j = 1$ and $k = 2$. Since X is well-formed $h | d_1$ and $h | d_2$. $P_0P_1P_2 \cap X$ is a finite number of points, all of type $\frac{1}{h}(a_3, a_4, a_5)$. As these are terminal, after renumbering, $h | a_3 + a_4$ and so $h | \alpha + a_5$. This is Condition (3). Condition (4) follows from the fact that $\mathrm{hcf}(h, a_3) = \mathrm{hcf}(h, a_4) = 1$.

So these conditions are necessary.

$\square$

III.6.6 Codimension 2 weighted 3-folds with trivial canonical bundle.

The 4 families of 3-fold codimension 2 quasismooth, well-formed complete intersections with at worst terminal singularities, $\omega_X \cong \mathcal{O}_X$ and $\sum a_i \leq 100$ are:

$X_{2,4}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1)$
$X_{3,3}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1)$
$X_{3,4}$ in $\mathbb{P}(1, 1, 1, 1, 1, 2)$
$X_{4,4}$ in $\mathbb{P}(1, 1, 1, 1, 2, 2)$

Again the above are all non-singular.

III.7 Canonically embedded weighted 3-folds.

III.7.1 Canonically embedded 3-fold weighted hypersurfaces.

There are 23 families of 3-fold weighted hypersurfaces with only terminal isolated quotient singularities with $\omega_X \cong \mathcal{O}_X(1)$ and $\sum a_i \leq 100$.

Hypersurface.	K_X^3	p_g	Singularities.
X_6 in $\mathbb{P}(1, 1, 1, 1, 1)$	6	5	
X_7 in $\mathbb{P}(1, 1, 1, 1, 2)$	7/2	4	$\frac{1}{2}(1, -1, 1)$
X_8 in $\mathbb{P}(1, 1, 1, 2, 2)$	2	3	$4 \times \frac{1}{2}(1, -1, 1)$

Hypersurface.	K_X^3	p_g	Singularities.
X_9 in $\mathbb{P}(1, 1, 1, 2, 3)$	3/2	3	$\frac{1}{2}(1, -1, 1)$
X_{10} in $\mathbb{P}(1, 1, 1, 1, 5)$	2	4	
X_{10} in $\mathbb{P}(1, 1, 2, 2, 3)$	5/6	2	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1)$
X_{12} in $\mathbb{P}(1, 1, 1, 2, 6)$	1	3	$2 \times \frac{1}{2}(1, -1, 1)$
X_{12} in $\mathbb{P}(1, 1, 2, 3, 4)$	1/2	2	$3 \times \frac{1}{2}(1, -1, 1)$
X_{12} in $\mathbb{P}(1, 2, 2, 3, 3)$	1/3	1	$6 \times \frac{1}{2}(1, -1, 1), 4 \times \frac{1}{3}(1, -1, 1)$
X_{14} in $\mathbb{P}(1, 1, 2, 2, 7)$	1/2	2	$7 \times \frac{1}{2}(1, -1, 1)$
X_{15} in $\mathbb{P}(1, 2, 3, 3, 5)$	1/6	1	$\frac{1}{2}(1, -1, 1), 5 \times \frac{1}{3}(1, -1, 1)$
X_{16} in $\mathbb{P}(1, 1, 2, 3, 8)$	1/3	2	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1)$
X_{16} in $\mathbb{P}(1, 2, 3, 4, 5)$	2/15	1	$4 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{18} in $\mathbb{P}(1, 2, 2, 3, 9)$	1/6	1	$9 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
X_{18} in $\mathbb{P}(2, 3, 3, 4, 5)$	1/20	0	$4 \times \frac{1}{2}(1, -1, 1), 6 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{20} in $\mathbb{P}(2, 3, 4, 5, 5)$	1/30	0	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), 4 \times \frac{1}{5}(1, -1, 2)$
X_{21} in $\mathbb{P}(1, 3, 4, 5, 7)$	1/20	1	$\frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{22} in $\mathbb{P}(1, 2, 3, 4, 11)$	1/12	1	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1)$
X_{28} in $\mathbb{P}(1, 3, 4, 5, 14)$	1/30	1	$\frac{1}{3}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1)$
X_{28} in $\mathbb{P}(3, 4, 5, 7, 8)$	1/120	0	$\frac{1}{3}(1, -1, 1), 3 \times \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{8}(1, -1, 3)$
X_{30} in $\mathbb{P}(2, 3, 4, 5, 15)$	1/60	0	$7 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 2)$
X_{40} in $\mathbb{P}(3, 4, 5, 7, 20)$	1/210	0	$\frac{1}{3}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 2), \frac{1}{7}(1, -1, 2)$
X_{46} in $\mathbb{P}(4, 5, 6, 7, 23)$	1/420	0	$\frac{1}{4}(1,-1,1), 3 \times \frac{1}{2}(1,-1,1), \frac{1}{5}(1,-1,2), \frac{1}{6}(1,-1,1), \frac{1}{7}(1,-1,3)$

This list was produced using the program *hyp.c* (see section III.10.7). In fact the program was run much further but produced no more examples. I conjecture that the lists in this section and in sections III.7.3, III.8.5, and III.8.6 are complete lists, and not limited by $\sum a_i \leq 100$.

III.7.2 Interesting Example. The family X_{46} in $\mathbb{P}(4, 5, 6, 7, 23)$ has p_g, P_2 and P_3 all zero. This is the closest, using weighted complete intersections, to answering the question posed in II.4.9.

III.7.3 Canonically embedded codimension 2 weighted 3-folds.

There are 59 families of 3-fold codimension 2 weighted complete intersections satisfying the conditions of Theorem III.6.4 with $\omega_X \cong \mathcal{O}_X(1)$ and $\sum a_i \leq 100$.

Complete Intersection	K_X^3	p_g	Singularities.
$X_{2,5}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1)$	10	6	
$X_{3,4}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1)$	12	6	
$X_{3,5}$ in $\mathbb{P}(1, 1, 1, 1, 1, 2)$	15/2	5	$\frac{1}{2}(1, -1, 1)$
$X_{4,4}$ in $\mathbb{P}(1, 1, 1, 1, 1, 2)$	8	5	
$X_{3,6}$ in $\mathbb{P}(1, 1, 1, 1, 2, 2)$	9/2	4	$3 \times \frac{1}{2}(1, -1, 1)$
$X_{4,5}$ in $\mathbb{P}(1, 1, 1, 1, 2, 2)$	5	4	$2 \times \frac{1}{2}(1, -1, 1)$

Complete Intersection	K_X^3	p_g	Singularities.
$X_{2,8}$ in $\mathbb{P}(1,1,1,1,1,4)$	4	5	
$X_{4,6}$ in $\mathbb{P}(1,1,1,1,2,3)$	4	4	
$X_{4,6}$ in $\mathbb{P}(1,1,1,2,2,2)$	3	3	$6\times\frac{1}{2}(1,-1,1)$
$X_{3,8}$ in $\mathbb{P}(1,1,1,1,2,4)$	3	4	$2\times\frac{1}{2}(1,-1,1)$
$X_{4,7}$ in $\mathbb{P}(1,1,1,2,2,3)$	7/3	3	$\frac{1}{3}(1,-1,1), 2\times\frac{1}{2}(1,-1,1)$
$X_{5,6}$ in $\mathbb{P}(1,1,1,2,2,3)$	5/2	3	$3\times\frac{1}{2}(1,-1,1)$
$X_{6,6}$ in $\mathbb{P}(1,1,1,2,3,3)$	2	3	
$X_{4,8}$ in $\mathbb{P}(1,1,2,2,2,3)$	4/3	2	$\frac{1}{3}(1,-1,1), 8\times\frac{1}{2}(1,-1,1)$
$X_{6,6}$ in $\mathbb{P}(1,1,2,2,2,3)$	3/2	2	$9\times\frac{1}{2}(1,-1,1)$
$X_{3,10}$ in $\mathbb{P}(1,1,1,2,2,5)$	3/2	3	$5\times\frac{1}{2}(1,-1,1)$
$X_{4,9}$ in $\mathbb{P}(1,1,2,2,3,3)$	1	2	$2\times\frac{1}{2}(1,-1,1), 3\times\frac{1}{3}(1,-1,1)$
$X_{6,7}$ in $\mathbb{P}(1,1,2,2,3,3)$	7/6	2	$3\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{4,10}$ in $\mathbb{P}(1,1,1,2,3,5)$	4/3	3	$\frac{1}{3}(1,-1,1)$
$X_{4,10}$ in $\mathbb{P}(1,1,2,2,2,5)$	1	2	$10\times\frac{1}{2}(1,-1,1)$
$X_{6,8}$ in $\mathbb{P}(1,1,2,2,3,4)$	1	2	$6\times\frac{1}{2}(1,-1,1)$
$X_{6,8}$ in $\mathbb{P}(1,2,2,2,3,3)$	2/3	1	$12\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{6,9}$ in $\mathbb{P}(1,1,2,3,3,4)$	3/4	2	$\frac{1}{4}(1,-1,1), \frac{1}{2}(1,-1,1)$
$X_{6,9}$ in $\mathbb{P}(1,2,2,3,3,3)$	1/2	1	$3\times\frac{1}{2}(1,-1,1), 6\times\frac{1}{3}(1,-1,1)$
$X_{4,12}$ in $\mathbb{P}(1,1,2,2,3,6)$	2/3	2	$4\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{6,10}$ in $\mathbb{P}(1,1,2,3,3,5)$	2/3	2	$2\times\frac{1}{3}(1,-1,1)$
$X_{6,10}$ in $\mathbb{P}(1,2,2,2,3,5)$	1/2	1	$15\times\frac{1}{2}(1,-1,1)$
$X_{6,10}$ in $\mathbb{P}(1,2,2,3,3,4)$	5/12	1	$\frac{1}{4}(1,-1,1), 7\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{4,14}$ in $\mathbb{P}(1,2,2,2,3,7)$	1/3	1	$\frac{1}{3}(1,-1,1), 14\times\frac{1}{2}(1,-1,1)$
$X_{6,12}$ in $\mathbb{P}(1,2,2,3,4,5)$	3/10	1	$\frac{1}{5}(1,-1,2), 9\times\frac{1}{2}(1,-1,1)$
$X_{8,10}$ in $\mathbb{P}(1,2,2,3,4,5)$	1/3	1	$\frac{1}{3}(1,-1,1), 10\times\frac{1}{2}(1,-1,1)$
$X_{6,12}$ in $\mathbb{P}(1,2,3,3,4,4)$	1/4	1	$3\times\frac{1}{2}(1,-1,1), 3\times\frac{1}{4}(1,-1,1)$
$X_{6,12}$ in $\mathbb{P}(2,2,3,3,3,4)$	1/6	0	$9\times\frac{1}{2}(1,-1,1), 8\times\frac{1}{3}(1,-1,1)$
$X_{6,13}$ in $\mathbb{P}(1,2,3,3,4,5)$	13/60	1	$\frac{1}{4}(1,-1,1), \frac{1}{5}(1,-1,2), \frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{9,10}$ in $\mathbb{P}(1,2,3,3,4,5)$	1/4	1	$\frac{1}{4}(1,-1,1), 2\times\frac{1}{2}(1,-1,1), 3\times\frac{1}{3}(1,-1,1)$
$X_{6,14}$ in $\mathbb{P}(1,2,2,3,4,7)$	1/4	1	$\frac{1}{4}(1,-1,1), 10\times\frac{1}{2}(1,-1,1)$
$X_{8,12}$ in $\mathbb{P}(1,2,3,4,4,5)$	1/5	1	$\frac{1}{5}(1,-1,1), 6\times\frac{1}{2}(1,-1,1)$
$X_{6,14}$ in $\mathbb{P}(2,2,2,3,3,7)$	1/6	0	$21\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{8,12}$ in $\mathbb{P}(2,2,3,3,4,5)$	2/15	0	$\frac{1}{5}(1,-1,2), 12\times\frac{1}{2}(1,-1,1), 4\times\frac{1}{3}(1,-1,1)$
$X_{6,15}$ in $\mathbb{P}(2,3,3,3,4,5)$	1/12	0	$\frac{1}{4}(1,-1,1), \frac{1}{2}(1,-1,1), 10\times\frac{1}{3}(1,-1,1)$
$X_{6,16}$ in $\mathbb{P}(1,2,3,3,4,8)$	1/6	1	$2\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1), 2\times\frac{1}{4}(1,-1,1)$
$X_{10,12}$ in $\mathbb{P}(1,2,3,4,5,6)$	1/6	1	$5\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
$X_{10,12}$ in $\mathbb{P}(2,2,3,4,5,5)$	1/10	0	$15\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{5}(1,-1,2)$
$X_{10,12}$ in $\mathbb{P}(2,3,3,4,4,5)$	1/12	0	$6\times\frac{1}{2}(1,-1,1), 4\times\frac{1}{3}(1,-1,1), 3\times\frac{1}{4}(1,-1,1)$

Complete Intersection	K_X^3	p_g	Singularities.
$X_{8,15}$ in $\mathbb{P}(2,3,3,4,5,5)$	1/15	0	$2\times\frac{1}{2}(1,-1,1)$, $5\times\frac{1}{3}(1,-1,1)$, $3\times\frac{1}{5}(1,-1,2)$
$X_{6,18}$ in $\mathbb{P}(1,2,3,3,5,9)$	2/15	1	$\frac{1}{5}(1,-1,2)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{6,18}$ in $\mathbb{P}(2,2,3,3,4,9)$	1/12	0	$\frac{1}{4}(1,-1,1)$, $13\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{10,14}$ in $\mathbb{P}(2,2,3,4,5,7)$	1/12	0	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $17\times\frac{1}{2}(1,-1,1)$
$X_{6,20}$ in $\mathbb{P}(1,2,3,4,5,10)$	1/10	1	$3\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$
$X_{12,14}$ in $\mathbb{P}(2,3,4,4,5,7)$	1/20	0	$\frac{1}{5}(1,-1,2)$, $9\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$
$X_{12,15}$ in $\mathbb{P}(1,3,4,5,6,7)$	1/14	1	$\frac{1}{7}(1,-1,2)$, $\frac{1}{3}(1,-1,1)$
$X_{10,18}$ in $\mathbb{P}(2,3,4,5,6,7)$	1/28	0	$\frac{1}{4}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$, $7\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(2,3,4,5,6,7)$	4/105	0	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,2)$, $8\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{8,22}$ in $\mathbb{P}(2,3,4,4,5,11)$	1/30	0	$\frac{1}{11}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $10\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1$
$X_{12,18}$ in $\mathbb{P}(2,3,4,5,6,9)$	1/30	0	$\frac{1}{5}(1,-1,1)$, $9\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{12,18}$ in $\mathbb{P}(3,4,4,5,6,7)$	3/140	0	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $3\times\frac{1}{4}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$
$X_{10,21}$ in $\mathbb{P}(3,4,5,5,6,7)$	1/60	0	$\frac{1}{4}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$
$X_{12,21}$ in $\mathbb{P}(3,4,5,6,7,7)$	1/70	0	$\frac{1}{5}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{7}(1,-1,2)$
$X_{12,28}$ in $\mathbb{P}(3,4,5,6,7,14)$	1/105	0	$\frac{1}{5}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{7}(1,-1,$

III.8 Q-Fano 3-folds.

In [R4, section 4.3] Reid conjectures that if X is a $\mathbb{Q}$-Fano 3-fold then $\mathcal{O}_X(-K_X)$ has a global section. This is false as shown by the following example:

III.8.1 Example.

The family $X_{12,14}$ in $\mathbb{P}(2,3,4,5,6,7)$ is an anticanonically embedded Fano 3-fold with only the following isolated terminal singularities: 1 of type $\frac{1}{5}(4,1,2)$, 2 of type $\frac{1}{3}(2,1,1)$ and 7 of type $\frac{1}{2}(1,1,1)$. Since it is well formed, $K_X^3 = -\frac{1}{30}$ and $\omega_X = \mathcal{O}_X(-1)$. Hence by the plurigenera formula (see section II.6) $K_X.c_2 = -\frac{101}{30} < 0$. However $\mathcal{O}_X(-K_X)$ has no global section.

The singularities are checked as follows. Let u, v, w, x, y and z be the homogeneous coordinates of weights 2, 3, 4, 5, 6 and 7 respectively. Let f, g be homogeneous polynomials of degrees 12 and 14 respectively. Then $X = (f,g)$ in $\mathbb{P} = \mathbb{P}(2,3,4,5,6,7)$.

Consider the vertices of the weighted projective space $\mathbb{P}$. Since $5 \nmid 12$ and $5 \nmid 14$, $P_3 \in X$. So

$$f = x^2u + \ldots$$
$$g = x^2w + \ldots$$

Thus $\{v, y, z\}$ are local coordinates around P_3, which is therefore a singularity of type $\frac{1}{5}(3,6,7)$, i.e. $\frac{1}{5}(4,1,2)$. There are no other vertices contained in X.

Consider the 1-dimensional loci of $\mathbb{P}$.

$P_0 P_2$: $h = \mathrm{hcf}(2, 4) = 2$ and

$$f = u^6 + w^3 + \ldots$$
$$g = u^7 + w^2 y + \ldots$$

So the local coordinates are $\{v, x, z\}$ and the singularities are of type $\frac{1}{2}(1, 1, 1)$. There are 3 such intersection points (by Lemma III.3.12 applied to X_6 in $\mathbb{P}(1, 2)$).

$P_0 P_4$: Likewise $h = \mathrm{hcf}(2, 6) = 2$ and

$$f = u^6 + y^2 + \ldots$$
$$g = u^7 + u^5 w + y^2 u + \ldots$$

$(f = 0)$ in $\mathbb{P}(1, 3)$ is 2 points by Lemma III.3.12. So there are 2 singularities of type $\frac{1}{2}(1, 1, 1)$ along $P_0 P_4$.

$P_2 P_4$: There is only 1 singularity of type $\frac{1}{2}(1, 1, 1)$ on this line.

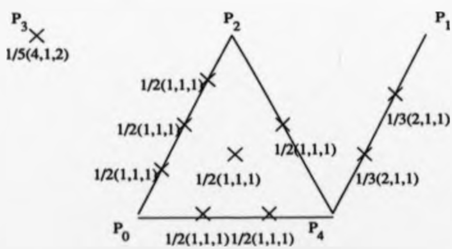
$P_1 P_4$: This time $h = \mathrm{hcf}(3, 6) = 3$ and

$$f = v^4 + y^2 + \ldots$$
$$g = v^4 u + y^2 u + \ldots$$

So there are 2 of type $\frac{1}{3}(1, -1, 1)$.

Consider the only singular 2-dimensional locus, $P_0 P_2 P_4$, of $\mathbb{P}$ where $h = \mathrm{hcf}(2, 4, 6) = 2$. By Lemma III.3.13, there are 7 intersection points (some of which have already been counted) of type $\frac{1}{2}(1, 1, 1)$.

The arrangement of singularities is shown below.



Experimentation leads to the following:

III.8.2 Conjecture: *Every weighted hypersurface* $\mathbf{Q}$*-Fano 3-fold* X*, with canonical singularities, has a global section of* ω_X^{-1}.

This is clear in one particular case.

III.8.3 Lemma: *Consider* X_d *in* $\mathbf{P}(a_0, ..., a_4)$ *be a family of* $\mathbf{Q}$*-Fano 3-folds with only isolated terminal singularities. Suppose also that* $a_0 \leq ... \leq a_4$ *and* $a_4 \nmid d$*. Then* ω_X^{-1} *has a global section.*

Proof. As $a_4 \nmid d$, the vertex P_4 is contained in X. The condition for a terminal singularity at P_i gives that there exists an a_m such that $a_4 | a_m + \alpha$. So $a_m = \mu a_4 + (-\alpha)$ for some integer μ. Since $\alpha < 0$ and $a_4 > a_m$, then $\mu \leq 0$. Thus $\deg(x_4^{(-\mu)} x_m) = -\alpha$ and so $\dim S_{-\alpha} \geq 1$. By [WPS, Theorem 1.4.1] $H^0(\omega_X^{-1}) \cong R_{-\alpha}$, where $R = \oplus_{n \geq 0} H^0(\mathcal{O}_X(n))$, and so ω_X^{-1} has a global section.

□

Notice that when $\alpha = -1$, there exists a generator x_i with $\deg(x_i) = 1$, i.e. $a_0 = 1$.

III.8.4 Lemma: *There is a bijection between the following:*

(i) *the set of families of quasismooth, well-formed weighted surface hypersurfaces* S_d *in* $\mathbf{P}(a_1, ..., a_4)$ *with only canonical singularities and trivial canonical class.*

(ii) *the set of families of quasismooth, well-formed weighted 3-folds hypersurfaces* X_d *in* $\mathbf{P}(1, a_1, ..., a_4)$ *with only canonical singularities and* $\omega_X \cong \mathcal{O}_X(-1)$.

Proof. Suppose that S_d in $\mathbf{P} = \mathbf{P}(a_1, ..., a_4)$ is a K3 surface, with at worst canonical singularities. By comparing the conditions in Theorems III.5.1 and III.6.1 it is clear that the conditions of the latter are satisfied for $X = X_d$ in $\mathbf{P}(1, a_1, ..., a_n)$. Thus X is quasismooth with at worst terminal singularities.

Conversely suppose X_d in $\mathbf{P}(1, a_1, ..., a_n)$ is quasismooth and has at worst terminal singularities. It can be seen from Theorems III.5.1 and III.6.1 that only condition (1*ii*) of Theorem III.4.1 needs proof (the others being either trivially satisfied or equivalent in both the surface and the 3-fold case).

Set $a_0 = 1$ and consider a_i for $i \neq 0$. Suppose that Condition (1*ii*) does not hold. So $a_i \nmid d - a_e$ for all $e = 1, ..., 4$. In particular $a_i \nmid d$. Thus $a_i | d - a_0$, i.e. $a_i | d - 1$. Since $a_i \nmid d$ then Theorem III.6.1 (1*iv*) gives that there exists an $m \neq 0, i$ such that $a_i | a_m - 1$. Hence $a_i | (d-1) - (a_m - 1)$, i.e. $a_i | d - a_m$, a contradiction. So $a_i | d - a_e$ for some $e \neq 0, i$ which is Condition (1*ii*) of Theorem III.5.1.

□

Each singularity on the K3 surface is of type $\frac{1}{r}(a, -a)$ for some r and a, with respect to some pair of the coordinates $x_1, ..., x_4$. Forming the corresponding $\mathbf{Q}$-Fano 3-fold results in an extra local coordinate x_0 at each singularity, which is thus of type $\frac{1}{r}(a, -a, 1)$. A similar result holds for higher codimensions.

This lemma gives a bijection between Reid's list of 95 families of weighted K3 surfaces (see section III.5.3 or [R4, section 4.5]) and the 95 families of weighted $\mathbf{Q}$-Fano 3-folds found by a computer search and listed below.

III.8.5 List of anti-canonically embedded (Q-Fano) weighted 3-folds.

The 95 families of weighted Q-Fano 3-folds, with $\alpha=-1$ and $\sum a_i \leq 100$, are listed below.

Hypersurface.	K_X^3	Singularities.
X_4 in $\mathbf{P}(1,1,1,1,1)$	-4	
X_5 in $\mathbf{P}(1,1,1,1,2)$	-5/2	$\frac{1}{2}(1,-1,1)$
X_6 in $\mathbf{P}(1,1,1,1,3)$	-2	
X_6 in $\mathbf{P}(1,1,1,2,2)$	-3/2	$3\times\frac{1}{2}(1,-1,1)$
X_7 in $\mathbf{P}(1,1,1,2,3)$	-7/6	$\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1)$
X_8 in $\mathbf{P}(1,1,1,2,4)$	-1	$2\times\frac{1}{2}(1,-1,1)$
X_8 in $\mathbf{P}(1,1,2,2,3)$	-2/3	$4\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1)$
X_9 in $\mathbf{P}(1,1,1,3,4)$	-3/4	$\frac{1}{4}(1,-1,1)$
X_9 in $\mathbf{P}(1,1,2,3,3)$	-1/2	$\frac{1}{2}(1,-1,1), 3\times\frac{1}{3}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,1,3,5)$	-2/3	$\frac{1}{3}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,2,2,5)$	-1/2	$5\times\frac{1}{2}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,2,3,4)$	-5/12	$2\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{4}(1,-1,1)$
X_{11} in $\mathbf{P}(1,1,2,3,5)$	-11/30	$\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{5}(1,-1,2)$
X_{12} in $\mathbf{P}(1,1,1,4,6)$	-1/2	$\frac{1}{2}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,2,3,6)$	-1/3	$2\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,2,4,5)$	-3/10	$3\times\frac{1}{2}(1,-1,1), \frac{1}{5}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,3,4,4)$	-1/4	$3\times\frac{1}{4}(1,-1,1)$
X_{12} in $\mathbf{P}(1,2,2,3,5)$	-1/5	$6\times\frac{1}{2}(1,-1,1), \frac{1}{5}(1,-1,2)$
X_{12} in $\mathbf{P}(1,2,3,3,4)$	-1/6	$3\times\frac{1}{2}(1,-1,1), 4\times\frac{1}{3}(1,-1,1)$
X_{13} in $\mathbf{P}(1,1,3,4,5)$	-13/60	$\frac{1}{3}(1,-1,1), \frac{1}{4}(1,-1,1), \frac{1}{5}(1,-1,1)$
X_{14} in $\mathbf{P}(1,1,2,4,7)$	-1/4	$3\times\frac{1}{2}(1,-1,1), \frac{1}{4}(1,-1,1)$
X_{14} in $\mathbf{P}(1,2,2,3,7)$	-1/6	$7\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1)$
X_{14} in $\mathbf{P}(1,2,3,4,5)$	-7/60	$3\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{4}(1,-1,1), \frac{1}{5}(1,-1,2)$
X_{15} in $\mathbf{P}(1,1,2,5,7)$	-3/14	$\frac{1}{2}(1,-1,1), \frac{1}{7}(1,-1,3)$
X_{15} in $\mathbf{P}(1,1,3,4,7)$	-5/28	$\frac{1}{4}(1,-1,1), \frac{1}{7}(1,-1,2)$
X_{15} in $\mathbf{P}(1,1,3,5,6)$	-1/6	$2\times\frac{1}{3}(1,-1,1), \frac{1}{6}(1,-1,1)$
X_{15} in $\mathbf{P}(1,2,3,5,5)$	-1/10	$\frac{1}{2}(1,-1,1), 3\times\frac{1}{5}(1,-1,2)$
X_{15} in $\mathbf{P}(1,3,3,4,5)$	-1/12	$5\times\frac{1}{3}(1,-1,1), \frac{1}{4}(1,-1,1)$
X_{16} in $\mathbf{P}(1,1,2,5,8)$	-1/5	$2\times\frac{1}{2}(1,-1,1), \frac{1}{5}(1,-1,2)$
X_{16} in $\mathbf{P}(1,1,3,4,8)$	-1/6	$\frac{1}{3}(1,-1,1), 2\times\frac{1}{4}(1,-1,1)$
X_{16} in $\mathbf{P}(1,1,4,5,6)$	-2/15	$\frac{1}{2}(1,-1,1), \frac{1}{5}(1,-1,1), \frac{1}{6}(1,-1,1)$
X_{16} in $\mathbf{P}(1,2,3,4,7)$	-2/21	$4\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{7}(1,-1,2)$
X_{17} in $\mathbf{P}(1,2,3,5,7)$	-17/210	$\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{5}(1,-1,2), \frac{1}{7}(1,-1,3)$
X_{18} in $\mathbf{P}(1,1,2,6,9)$	-1/6	$3\times\frac{1}{2}(1,-1,1), \frac{1}{3}(1,-1,1)$

Hypersurface.	K_X^3	Singularities.
X_{18} in $\mathbb{P}(1, 1, 3, 5, 9)$	-2/15	$2 \times \frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 1)$
X_{18} in $\mathbb{P}(1, 1, 4, 6, 7)$	-3/28	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{7}(1, -1, 1)$
X_{18} in $\mathbb{P}(1, 2, 3, 4, 9)$	-1/12	$4 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1)$
X_{18} in $\mathbb{P}(1, 2, 3, 5, 8)$	-3/40	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{8}(1, -1, 3)$
X_{18} in $\mathbb{P}(1, 3, 4, 5, 6)$	-1/20	$3 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1)$
X_{19} in $\mathbb{P}(1, 3, 4, 5, 7)$	-19/420	$\frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{7}(1, -1, 2)$
X_{20} in $\mathbb{P}(1, 1, 4, 5, 10)$	-1/10	$\frac{1}{2}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 1)$
X_{20} in $\mathbb{P}(1, 2, 3, 5, 10)$	-1/15	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 2)$
X_{20} in $\mathbb{P}(1, 2, 4, 5, 9)$	-1/18	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{9}(1, -1, 2)$
X_{20} in $\mathbb{P}(1, 2, 5, 6, 7)$	-1/21	$3 \times \frac{1}{2}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
X_{20} in $\mathbb{P}(1, 3, 4, 5, 8)$	-1/24	$\frac{1}{3}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1), \frac{1}{8}(1, -1, 3)$
X_{21} in $\mathbb{P}(1, 1, 3, 7, 10)$	-1/10	$\frac{1}{10}(1, -1, 3)$
X_{21} in $\mathbb{P}(1, 1, 5, 7, 8)$	-3/40	$\frac{1}{5}(1, -1, 2), \frac{1}{8}(1, -1, 1)$
X_{21} in $\mathbb{P}(1, 2, 3, 7, 9)$	-1/18	$\frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1), \frac{1}{9}(1, -1, 4)$
X_{21} in $\mathbb{P}(1, 3, 5, 6, 7)$	-1/30	$3 \times \frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{6}(1, -1, 1)$
X_{22} in $\mathbb{P}(1, 1, 3, 7, 11)$	-2/21	$\frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 2)$
X_{22} in $\mathbb{P}(1, 1, 4, 6, 11)$	-1/12	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{6}(1, -1, 1)$
X_{22} in $\mathbb{P}(1, 2, 4, 5, 11)$	-1/20	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 1)$
X_{24} in $\mathbb{P}(1, 1, 3, 8, 12)$	-1/12	$2 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1)$
X_{24} in $\mathbb{P}(1, 1, 6, 8, 9)$	-1/18	$\frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{9}(1, -1, 1)$
X_{24} in $\mathbb{P}(1, 2, 3, 7, 12)$	-1/21	$2 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
X_{24} in $\mathbb{P}(1, 2, 3, 8, 11)$	-1/22	$3 \times \frac{1}{2}(1, -1, 1), \frac{1}{11}(1, -1, 4)$
X_{24} in $\mathbb{P}(1, 3, 4, 5, 12)$	-1/30	$2 \times \frac{1}{3}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{24} in $\mathbb{P}(1, 3, 4, 7, 10)$	-1/35	$\frac{1}{2}(1, -1, 1), \frac{1}{7}(1, -1, 2), \frac{1}{10}(1, -1, 3)$
X_{24} in $\mathbb{P}(1, 3, 6, 7, 8)$	-1/42	$4 \times \frac{1}{3}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{7}(1, -1, 1)$
X_{24} in $\mathbb{P}(1, 4, 5, 6, 9)$	-1/45	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{9}(1, -1, 2)$
X_{25} in $\mathbb{P}(1, 4, 5, 7, 9)$	-5/252	$\frac{1}{4}(1, -1, 1), \frac{1}{7}(1, -1, 3), \frac{1}{9}(1, -1, 2)$
X_{26} in $\mathbb{P}(1, 1, 5, 7, 13)$	-2/35	$\frac{1}{5}(1, -1, 2), \frac{1}{7}(1, -1, 1)$
X_{26} in $\mathbb{P}(1, 2, 3, 8, 13)$	-1/24	$3 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{8}(1, -1, 3)$
X_{26} in $\mathbb{P}(1, 2, 5, 6, 13)$	-1/30	$4 \times \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{6}(1, -1, 1)$
X_{27} in $\mathbb{P}(1, 2, 5, 9, 11)$	-3/110	$\frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{11}(1, -1, 5)$
X_{27} in $\mathbb{P}(1, 5, 6, 7, 9)$	-1/70	$\frac{1}{5}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
X_{28} in $\mathbb{P}(1, 1, 4, 9, 14)$	-1/18	$\frac{1}{2}(1, -1, 1), \frac{1}{9}(1, -1, 2)$
X_{28} in $\mathbb{P}(1, 3, 4, 7, 14)$	-1/42	$\frac{1}{3}(1, -1, 1), \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{7}(1, -1, 2)$
X_{28} in $\mathbb{P}(1, 4, 6, 7, 11)$	-1/66	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{11}(1, -1, 3)$
X_{30} in $\mathbb{P}(1, 1, 4, 10, 15)$	-1/20	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1)$
X_{30} in $\mathbb{P}(1, 1, 6, 8, 15)$	-1/24	$\frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{8}(1, -1, 1)$

Hypersurface.	K_X^3	Singularities.
X_{30} in $\mathbf{P}(1, 2, 3, 10, 15)$	-1/30	$3 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{30} in $\mathbf{P}(1, 2, 6, 7, 15)$	-1/42	$5 \times \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 1)$
X_{30} in $\mathbf{P}(1, 3, 4, 10, 13)$	-1/52	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{13}(1, -1, 4)$
X_{30} in $\mathbf{P}(1, 4, 5, 6, 15)$	-1/60	$\frac{1}{4}(1, -1, 1), 2 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 1), \frac{1}{3}(1, -1, 1)$
X_{30} in $\mathbf{P}(1, 5, 6, 8, 11)$	-1/88	$\frac{1}{2}(1, -1, 1), \frac{1}{8}(1, -1, 3), \frac{1}{11}(1, -1, 2)$
X_{32} in $\mathbf{P}(1, 2, 5, 9, 16)$	-1/45	$2 \times \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{9}(1, -1, 4)$
X_{32} in $\mathbf{P}(1, 4, 5, 7, 16)$	-1/70	$2 \times \frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
X_{33} in $\mathbf{P}(1, 3, 5, 11, 14)$	-1/70	$\frac{1}{5}(1, -1, 1), \frac{1}{14}(1, -1, 5)$
X_{34} in $\mathbf{P}(1, 3, 4, 10, 17)$	-1/60	$\frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{10}(1, -1, 3)$
X_{34} in $\mathbf{P}(1, 4, 6, 7, 17)$	-1/84	$\frac{1}{4}(1, -1, 1), 2 \times \frac{1}{2}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{7}(1, -1, 2)$
X_{36} in $\mathbf{P}(1, 1, 5, 12, 18)$	-1/30	$\frac{1}{5}(1, -1, 2), \frac{1}{6}(1, -1, 1)$
X_{36} in $\mathbf{P}(1, 3, 4, 11, 18)$	-1/66	$2 \times \frac{1}{3}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{11}(1, -1, 3)$
X_{36} in $\mathbf{P}(1, 7, 8, 9, 12)$	-1/168	$\frac{1}{7}(1, -1, 3), \frac{1}{8}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{3}(1, -1, 1)$
X_{38} in $\mathbf{P}(1, 3, 5, 11, 19)$	-2/165	$\frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{11}(1, -1, 4)$
X_{38} in $\mathbf{P}(1, 5, 6, 8, 19)$	-1/120	$\frac{1}{5}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{8}(1, -1, 3)$
X_{40} in $\mathbf{P}(1, 5, 7, 8, 20)$	-1/140	$2 \times \frac{1}{5}(1, -1, 2), \frac{1}{7}(1, -1, 1), \frac{1}{4}(1, -1, 1)$
X_{42} in $\mathbf{P}(1, 1, 6, 14, 21)$	-1/42	$\frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 1)$
X_{42} in $\mathbf{P}(1, 2, 5, 14, 21)$	-1/70	$3 \times \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
X_{42} in $\mathbf{P}(1, 3, 4, 14, 21)$	-1/84	$2 \times \frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{7}(1, -1, 2)$
X_{44} in $\mathbf{P}(1, 4, 5, 13, 22)$	-1/130	$\frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{13}(1, -1, 3)$
X_{48} in $\mathbf{P}(1, 3, 5, 16, 24)$	-1/120	$2 \times \frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 1), \frac{1}{8}(1, -1, 3)$
X_{50} in $\mathbf{P}(1, 7, 8, 10, 25)$	-1/280	$\frac{1}{7}(1, -1, 2), \frac{1}{8}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2)$
X_{54} in $\mathbf{P}(1, 4, 5, 18, 27)$	-1/180	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{9}(1, -1, 2)$
X_{66} in $\mathbf{P}(1, 5, 6, 22, 33)$	-1/330	$\frac{1}{5}(1, -1, 2), \frac{1}{2}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{11}(1, -1, 2)$

III.8.6 Codimension 2 Q-Fano weighted complete intersections.

There are 85 codimension 2 $\mathbf{Q}$-Fano weighted complete intersections which satisfy the conditions of Theorem III.6.4, $\alpha = -1$ and $\sum a_i \leq 100$.

Complete intersection	K_X^3	Singularities.
$X_{2,3}$ in $\mathbf{P}(1, 1, 1, 1, 1, 1)$	-6	
$X_{3,3}$ in $\mathbf{P}(1, 1, 1, 1, 1, 2)$	-9/2	$\frac{1}{2}(1, -1, 1)$
$X_{3,4}$ in $\mathbf{P}(1, 1, 1, 1, 2, 2)$	-3	$2 \times \frac{1}{2}(1, -1, 1)$
$X_{4,4}$ in $\mathbf{P}(1, 1, 1, 1, 2, 3)$	-8/3	$\frac{1}{3}(1, -1, 1)$
$X_{4,4}$ in $\mathbf{P}(1, 1, 1, 2, 2, 2)$	-2	$4 \times \frac{1}{2}(1, -1, 1)$
$X_{4,5}$ in $\mathbf{P}(1, 1, 1, 2, 2, 3)$	-5/3	$\frac{1}{3}(1, -1, 1), 2 \times \frac{1}{2}(1, -1, 1)$
$X_{4,6}$ in $\mathbf{P}(1, 1, 1, 2, 3, 3)$	-4/3	$2 \times \frac{1}{3}(1, -1, 1)$
$X_{4,6}$ in $\mathbf{P}(1, 1, 2, 2, 2, 3)$	-1	$6 \times \frac{1}{2}(1, -1, 1)$
$X_{5,6}$ in $\mathbf{P}(1, 1, 1, 2, 3, 4)$	-5/4	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1)$

Complete intersection	K_X^3	Singularities.
$X_{5,6}$ in $\mathbb{P}(1, 1, 2, 2, 3, 3)$	-5/6	$3 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{6,6}$ in $\mathbb{P}(1, 1, 1, 2, 3, 5)$	-6/5	$\frac{1}{5}(1, -1, 2)$
$X_{6,6}$ in $\mathbb{P}(1, 1, 2, 2, 3, 4)$	-3/4	$\frac{1}{4}(1, -1, 1), 4 \times \frac{1}{2}(1, -1, 1)$
$X_{6,6}$ in $\mathbb{P}(1, 1, 2, 3, 3, 3)$	-2/3	$4 \times \frac{1}{3}(1, -1, 1)$
$X_{6,6}$ in $\mathbb{P}(1, 2, 2, 2, 3, 3)$	-1/2	$9 \times \frac{1}{2}(1, -1, 1)$
$X_{6,7}$ in $\mathbb{P}(1, 1, 2, 2, 3, 5)$	-7/10	$\frac{1}{5}(1, -1, 2), 3 \times \frac{1}{2}(1, -1, 1)$
$X_{6,7}$ in $\mathbb{P}(1, 1, 2, 3, 3, 4)$	-7/12	$\frac{1}{4}(1, -1, 1), \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{6,8}$ in $\mathbb{P}(1, 1, 1, 3, 4, 5)$	-4/5	$\frac{1}{5}(1, -1, 1)$
$X_{6,8}$ in $\mathbb{P}(1, 1, 2, 3, 3, 5)$	-8/15	$\frac{1}{5}(1, -1, 2), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{6,8}$ in $\mathbb{P}(1, 1, 2, 3, 4, 4)$	-1/2	$2 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1)$
$X_{6,8}$ in $\mathbb{P}(1, 2, 2, 3, 3, 4)$	-1/3	$6 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{6,9}$ in $\mathbb{P}(1, 1, 2, 3, 4, 5)$	-9/20	$\frac{1}{4}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{2}(1, -1, 1)$
$X_{7,8}$ in $\mathbb{P}(1, 1, 2, 3, 4, 5)$	-7/15	$\frac{1}{3}(1, -1, 1), \frac{1}{5}(1, -1, 1), 2 \times \frac{1}{2}(1, -1, 1)$
$X_{6,10}$ in $\mathbb{P}(1, 1, 2, 3, 5, 5)$	-2/5	$2 \times \frac{1}{5}(1, -1, 2)$
$X_{6,10}$ in $\mathbb{P}(1, 2, 2, 3, 4, 5)$	-1/4	$\frac{1}{4}(1, -1, 1), 7 \times \frac{1}{2}(1, -1, 1)$
$X_{8,9}$ in $\mathbb{P}(1, 1, 2, 3, 4, 7)$	-3/7	$\frac{1}{7}(1, -1, 2), 2 \times \frac{1}{2}(1, -1, 1)$
$X_{8,9}$ in $\mathbb{P}(1, 1, 3, 4, 4, 5)$	-3/10	$\frac{1}{5}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1)$
$X_{8,9}$ in $\mathbb{P}(1, 2, 3, 3, 4, 5)$	-1/5	$\frac{1}{5}(1, -1, 2), 2 \times \frac{1}{2}(1, -1, 1), 3 \times \frac{1}{3}(1, -1, 1)$
$X_{8,10}$ in $\mathbb{P}(1, 1, 2, 3, 5, 7)$	-8/21	$\frac{1}{3}(1, -1, 1), \frac{1}{7}(1, -1, 3)$
$X_{8,10}$ in $\mathbb{P}(1, 1, 2, 4, 5, 6)$	-1/3	$\frac{1}{6}(1, -1, 1), 3 \times \frac{1}{2}(1, -1, 1)$
$X_{8,10}$ in $\mathbb{P}(1, 1, 3, 4, 5, 5)$	-4/15	$\frac{1}{3}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 1)$
$X_{8,10}$ in $\mathbb{P}(1, 2, 3, 4, 4, 5)$	-1/6	$\frac{1}{3}(1, -1, 1), 4 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{4}(1, -1, 1)$
$X_{9,10}$ in $\mathbb{P}(1, 1, 2, 3, 5, 8)$	-3/8	$\frac{1}{8}(1, -1, 3), \frac{1}{2}(1, -1, 1)$
$X_{9,10}$ in $\mathbb{P}(1, 1, 3, 4, 5, 6)$	-1/4	$\frac{1}{4}(1, -1, 1), \frac{1}{6}(1, -1, 1), \frac{1}{3}(1, -1, 1)$
$X_{9,10}$ in $\mathbb{P}(1, 2, 2, 3, 5, 7)$	-3/14	$\frac{1}{7}(1, -1, 3), 5 \times \frac{1}{2}(1, -1, 1)$
$X_{9,10}$ in $\mathbb{P}(1, 2, 3, 4, 5, 5)$	-3/20	$\frac{1}{4}(1, -1, 1), 2 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 2)$
$X_{8,12}$ in $\mathbb{P}(1, 1, 3, 4, 5, 7)$	-8/35	$\frac{1}{5}(1, -1, 1), \frac{1}{7}(1, -1, 2)$
$X_{8,12}$ in $\mathbb{P}(1, 2, 3, 4, 5, 6)$	-2/15	$\frac{1}{5}(1, -1, 1), 4 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{9,12}$ in $\mathbb{P}(1, 2, 3, 4, 5, 7)$	-9/70	$\frac{1}{5}(1, -1, 2), \frac{1}{7}(1, -1, 2), 3 \times \frac{1}{2}(1, -1, 1)$
$X_{10,11}$ in $\mathbb{P}(1, 2, 3, 4, 5, 7)$	-11/84	$\frac{1}{3}(1, -1, 1), \frac{1}{4}(1, -1, 1), \frac{1}{7}(1, -1, 3), 2 \times \frac{1}{2}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 1, 3, 4, 5, 9)$	-2/9	$\frac{1}{9}(1, -1, 2), \frac{1}{3}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 1, 3, 5, 6, 7)$	-4/21	$\frac{1}{7}(1, -1, 1), 2 \times \frac{1}{3}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 1, 4, 5, 6, 6)$	-1/6	$\frac{1}{2}(1, -1, 1), 2 \times \frac{1}{6}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 2, 3, 4, 5, 8)$	-1/8	$\frac{1}{8}(1, -1, 3), 3 \times \frac{1}{2}(1, -1, 1), \frac{1}{4}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 2, 3, 5, 5, 7)$	-4/35	$\frac{1}{7}(1, -1, 3), 2 \times \frac{1}{5}(1, -1, 2)$
$X_{10,12}$ in $\mathbb{P}(1, 2, 4, 5, 5, 6)$	-1/10	$5 \times \frac{1}{2}(1, -1, 1), 2 \times \frac{1}{5}(1, -1, 1)$
$X_{10,12}$ in $\mathbb{P}(1, 3, 3, 4, 5, 7)$	-2/21	$\frac{1}{7}(1, -1, 2), 4 \times \frac{1}{3}(1, -1, 1)$

Complete intersection	K_X^3	Singularities.
$X_{10,12}$ in $\mathbb{P}(1,3,4,4,5,6)$	-1/12	$2\times\frac{1}{3}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{11,12}$ in $\mathbb{P}(1,1,4,5,6,7)$	-11/70	$\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
$X_{10,14}$ in $\mathbb{P}(1,1,2,5,7,9)$	-2/9	$\frac{1}{9}(1,-1,4)$
$X_{10,14}$ in $\mathbb{P}(1,2,3,5,7,7)$	-2/21	$\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{7}(1,-1,3)$
$X_{10,14}$ in $\mathbb{P}(1,2,4,5,6,7)$	-1/12	$\frac{1}{4}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $5\times\frac{1}{2}(1,-1,1)$
$X_{10,15}$ in $\mathbb{P}(1,2,3,5,7,8)$	-5/56	$\frac{1}{7}(1,-1,3)$, $\frac{1}{8}(1,-1,3)$, $\frac{1}{2}(1,-1,1)$
$X_{12,13}$ in $\mathbb{P}(1,3,4,5,6,7)$	-13/210	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $2\times\frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(1,1,3,4,7,11)$	-2/11	$\frac{1}{11}(1,-1,3)$
$X_{12,14}$ in $\mathbb{P}(1,1,4,6,7,8)$	-1/8	$\frac{1}{8}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(1,2,3,4,7,10)$	-1/10	$\frac{1}{10}(1,-1,3)$, $4\times\frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(1,2,3,5,7,9)$	-4/45	$\frac{1}{5}(1,-1,2)$, $\frac{1}{9}(1,-1,4)$, $\frac{1}{3}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(1,3,4,5,7,7)$	-2/35	$\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{7}(1,-1,2)$
$X_{12,14}$ in $\mathbb{P}(1,4,4,5,6,7)$	-1/20	$\frac{1}{5}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbb{P}(2,3,4,5,6,7)$	-1/30	$\frac{1}{5}(1,-1,2)$, $7\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{12,15}$ in $\mathbb{P}(1,1,4,5,6,11)$	-3/22	$\frac{1}{11}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$
$X_{12,15}$ in $\mathbb{P}(1,3,4,5,6,9)$	-1/18	$\frac{1}{9}(1,-1,2)$, $3\times\frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{12,15}$ in $\mathbb{P}(1,3,4,5,7,8)$	-3/56	$\frac{1}{7}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$, $\frac{1}{4}(1,-1,1)$
$X_{12,16}$ in $\mathbb{P}(1,2,5,6,7,8)$	-2/35	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,1)$, $4\times\frac{1}{2}(1,-1,1)$
$X_{14,15}$ in $\mathbb{P}(1,2,3,5,7,12)$	-1/12	$\frac{1}{12}(1,-1,5)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{14,15}$ in $\mathbb{P}(1,2,5,6,7,9)$	-1/18	$\frac{1}{6}(1,-1,1)$, $\frac{1}{9}(1,-1,4)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{14,15}$ in $\mathbb{P}(1,3,4,5,7,10)$	-1/20	$\frac{1}{4}(1,-1,1)$, $\frac{1}{10}(1,-1,3)$, $\frac{1}{5}(1,-1,2)$
$X_{14,15}$ in $\mathbb{P}(1,3,5,6,7,8)$	-1/24	$\frac{1}{6}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{14,16}$ in $\mathbb{P}(1,1,5,7,8,9)$	-4/45	$\frac{1}{5}(1,-1,2)$, $\frac{1}{9}(1,-1,1)$
$X_{14,16}$ in $\mathbb{P}(1,3,4,5,7,11)$	-8/165	$\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{11}(1,-1,3)$
$X_{14,16}$ in $\mathbb{P}(1,4,5,6,7,8)$	-1/30	$\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1)$
$X_{15,16}$ in $\mathbb{P}(1,2,3,5,8,13)$	-1/13	$\frac{1}{13}(1,-1,5)$, $2\times\frac{1}{2}(1,-1,1)$
$X_{15,16}$ in $\mathbb{P}(1,3,4,5,8,11)$	-1/22	$\frac{1}{11}(1,-1,4)$, $2\times\frac{1}{2}(1,-1,1)$
$X_{14,18}$ in $\mathbb{P}(1,2,3,7,9,11)$	-2/33	$\frac{1}{11}(1,-1,5)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{14,18}$ in $\mathbb{P}(1,2,6,7,8,9)$	-1/24	$\frac{1}{8}(1,-1,1)$, $5\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{12,20}$ in $\mathbb{P}(1,4,5,6,7,10)$	-1/35	$\frac{1}{7}(1,-1,2)$, $2\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,1)$
$X_{16,18}$ in $\mathbb{P}(1,1,6,8,9,10)$	-1/15	$\frac{1}{10}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{16,18}$ in $\mathbb{P}(1,4,6,7,8,9)$	-1/42	$\frac{1}{7}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{18,20}$ in $\mathbb{P}(1,4,5,6,9,14)$	-1/42	$\frac{1}{14}(1,-1,3)$, $2\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{18,20}$ in $\mathbb{P}(1,4,5,7,9,13)$	-2/91	$\frac{1}{7}(1,-1,3)$, $\frac{1}{13}(1,-1,3)$
$X_{18,20}$ in $\mathbb{P}(1,5,6,7,9,11)$	-4/231	$\frac{1}{7}(1,-1,3)$, $\frac{1}{11}(1,-1,2)$, $\frac{1}{3}(1,-1,1)$
$X_{18,22}$ in $\mathbb{P}(1,2,5,9,11,13)$	-2/65	$\frac{1}{5}(1,-1,1)$, $\frac{1}{13}(1,-1,6)$
$X_{20,21}$ in $\mathbb{P}(1,3,4,7,10,17)$	-1/34	$\frac{1}{17}(1,-1,5)$, $\frac{1}{2}(1,-1,1)$

Complete intersection	K_X^3	Singularities.
$X_{18,30}$ in $\mathbb{P}(1,6,8,9,10,15)$	-1/120	$\frac{1}{8}(1,-1,1), 2\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1), \frac{1}{5}(1,-1,1)$
$X_{24,30}$ in $\mathbb{P}(1,8,9,10,12,15)$	-1/180	$\frac{1}{9}(1,-1,1), \frac{1}{2}(1,-1,1), \frac{1}{4}(1,-1,1), \frac{1}{3}(1,-1,1), \frac{1}{5}(1,-1,2)$

III.9 The Reid Table Method

Consider a complete intersection $X_{d_1,\ldots,d_c}$ in $\mathbb{P}(a_0,\ldots,a_n)$. The Poincaré series (see [WPS, section 3.4] and compare [A&M, 11.1]) corresponding to the coordinate ring R of X is:

$$\mathcal{P}(t) = \sum_{n=0}^{\infty} h^0(X, \mathcal{O}_X(n))t^n$$
$$= \frac{\prod_{i=1}^{i=c}(1-t^{d_i})}{\prod_{i=0}^{i=n}(1-t^{a_i})}$$

Moreover if $\omega_X \cong \mathcal{O}_X(1)$ then $\mathcal{P}(t) = \sum_{n=0}^{\infty} P_n(X)t^n$, where $P_n(X)$ are the plurigenera of X. In the case of a Q-Fano 3-fold with $\omega_X \cong \mathcal{O}_X(-1)$ then $\mathcal{P}(t) = \sum_{n=0}^{\infty} P_{-n}(X)t^n$, where $P_{-n}(X)$ are the anti-plurigenera of X.

III.9.1 Example. X_6 in $\mathbb{P}^4$ has Poincaré series
$$\mathcal{P}(t) = \frac{(1-t^6)}{(1-t)^5}.$$
So $p_g = 1, P_2 = 5, P_3 = 15$, etc..

III.9.2 Question. Given a list of plurigenera (which could arise from a record of pluridata) does there exist a complete intersection with $\omega_X \cong \mathcal{O}_X(\pm 1)$?

The following lemma due to Reid helps answer the above.

III.9.3 Lemma: *Given a sequence $p_0 = 1, p_1, p_2, \ldots$ such that*
$$\sum_{i=0}^{\infty} p_i t^i = \frac{\prod_{i=1}^{i=c}(1-t^{d_i})}{\prod_{i=0}^{i=n}(1-t^{a_i})}$$

for some $\{d_i, a_i\}$. Then these $\{d_i, a_i\}$ are unique up to $a_i \neq d_j$ and are determinable.

Proof. The following is a constructive proof. Let $q_i^0 = p_i$. So

$$\sum_{i=0}^{\infty} q_i^0 t^i = \frac{\Pi(1-t^{d_i})}{\Pi(1-t^{a_i})}.$$

Without loss of generality assume that $d_c \geq \ldots \geq d_1$ and $a_n \geq \ldots \geq a_0$. Clearly we may assume $a_0 \neq d_1$ or else these two terms would cancel. There are two cases:

(i) $a_0 < d_1$.

Let a_0 occur with multiplicity μ. Then $\mathcal{P}(t) = 1 + \mu t^{a_0}$ + higher order terms. So the first non-zero q_i^0 is $q_{a_0}^0 = \mu > 0$. Define $q_i^1 = q_i^0 - q_{i-a_0}^0$, where $q_i^0 = 0$ if $i < 0$. Then $q_{a_0}^1 = q_{a_0}^0 - 1$. Thus

$$\begin{aligned}\sum_{i=0}^{\infty} q_i^1 t^i &= \sum_{i=0}^{\infty} (q_i^0 - q_{i-a_0}^0) t^i \\ &= (1-t^{a_0}) \sum_{i=0}^{\infty} q_i^0 t^i \\ &= \frac{\prod_{i=1}^{c} (1-t^{d_i})}{\prod_{i=1}^{n} (1-t^{a_i})}.\end{aligned}$$

This then involves one less a_i.

(ii) $d_1 < a_0$.

Let d_1 occur with multiplicity μ. Then $\mathcal{P}(t) = 1 - \mu t^{d_1}$ + higher order terms. So the first non-zero q_i^0 is $q_{d_1}^0 = -\mu < 0$. Define $q_i^1 = q_i^0 + q_{i-d_1}^1$, for $i = 1, 2, \ldots$ where $q_i^1 = 0$ if $i < 0$. This corresponds to:

$$\begin{aligned}\sum_{i=0}^{\infty} q_i^1 t^i &= \sum_{i=0}^{\infty} (q_i^0 + q_{i-d_1}^1) t^i \\ &= \sum_{i=0}^{\infty} (q_i^0 + q_{i-d_1}^0 + q_{i-2d_1}^0 + \ldots) t^i \\ &= \frac{\prod_{i=2}^{c} (1-t^{d_i})}{\prod_{i=0}^{n} (1-t^{a_i})}.\end{aligned}$$

This involves one less d_i.

Repetition of the above steps clearly terminates when

$$\sum_{i=0}^{\infty} q_i^b t^i = 1$$

By induction on the number of $\{a_i\}$ and $\{d_j\}$ it is clear that the process uniquely determines the a_i and d_j.

$\square$

III.9.4 Note. So the proof of the above Lemma allows us to construct a weighted complete intersection from a list of 'plurigenera'. If the lists q_i^m of integers are written as the columns of a table, the process is clearly defined. The integers at the head of each column keep track of the a_i and $-d_i$.

III.9.5 Example. Consider the record of pluridata $K^3 = \frac{1}{6}$, $\chi = 1$, $p_g = 0$, 9 singularities of type $\frac{1}{2}(1, 1, 1)$ and 8 singularities of type $\frac{1}{3}(2, 1, 1)$. The table obtained is the following:

n	P_n	(2)	(2)	(3)	(3)	(3)	(4)	(-6)	(-12)
0	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0
3	3	3	3	2	1	0	0	0	0
4	4	2	1	1	1	1	0	0	0
5	6	3	0	0	0	0	0	0	0
6	11	7	5	2	0	-1	-1	0	0
7	12	6	3	2	1	0	0	0	0
8	19	8	1	1	1	1	0	0	0
9	25	13	7	2	0	0	0	0	0
10	32	13	5	2	0	-1	0	0	0
11	41	16	3	2	1	0	0	0	0
12	54	22	9	2	0	0	-1	-1	0
13	64	23	7	2	0	0	0	0	0
14	81	27	5	2	0	-1	0	0	0
15	98	34	11	2	0	0	0	0	0
16	117	36	9	2	0	0	0	0	0
17	139	41	7	2	0	0	0	0	0
18	166	49	13	2	0	0	1	0	0
19	191	52	11	2	0	0	0	0	0
20	224	58	9	2	0	0	0	0	0

This gives $X_{6,12}$ in $\mathbb{P}(2, 2, 3, 3, 3, 4)$, which has the above record.

III.9.6 Note. Of course this method cannot tell the difference between X_6 in $\mathbb{P}(1, 1, 1, 2)$ and the example of V. Iliev $X_{3,6}$ in $\mathbb{P}(1, 1, 1, 2, 3)$, in which the cubic relation does not involve the degree 3 generator.

III.9.7 Warning. Although in general it is clear when this process stops, it is not clear when it is worth continuing with a particular list of integers.

III.9.8 The analysis.

This process is basically the same as that in section III.4.4 on the coordinate ring $R = \oplus_m R_m$. Starting from the dimensions of each R_m the degrees of the generators and relations can be found. At each stage it is assumed that the monomials are linearly independent unless

(i) there already exist relations of a lower degree, or

(ii) a relation is forced by the dimension not being large enough.

For the above example we have the following analysis:

Degree	Dimension	Monomials	
0	1	1	
1	0	$\varnothing$	
2	2	$x_0, x_1.$	
3	3	$y_0, y_1, y_2.$	
4	4	$x_0^2, x_0x_1, x_1^2, z.$	
5	6	$x_0y_0, x_0y_1, x_0y_2, x_1y_0, x_1y_1, x_1y_2.$	
6	11	$x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, y_0^2, y_0y_1, y_0y_2,$ $y_1^2, y_1y_2, y_2^2, x_0z, x_1z.$	1 relation.

If this calculation is continued only one more relation is found, which is of degree 12

III.9.9 Canonical 3-fold complete intersections.

The formula:

$$P_2 = \tfrac{1}{2}K_X^3 - 3(1-p_g) + l(2)$$

limits the value of p_g (since $K_X^3 > 0$) and defines K_X^3 in terms of a particular basket of singularities and P_2.

III.9.10 Q-Fano complete intersections.

The formula:

$$P_{-1} = -\tfrac{1}{2}K_X^3 + 3 - l(2)$$

defines K_X^3 in terms of a particular basket of singularities and P_{-1}.

III.9.11 The search. The search through all combinations of $P \geq 0$ ($P_2 = P$ for canonical 3-folds and $P_{-1} = P$ for the Fano case) and baskets will give every possible list of plurigenera (respectively anti-plurigenera). Hence a list of canonically (respectively anti-canonically) embedded complete intersections can be found. Of course this is not a finite search.

The program *totalsearch.c*, which performs this search, is found in the appendix (see section III.10.9). Let Q_i for $i = 0, 1, \ldots$ be a list of the types of 3-fold cyclic quotient singularity $\dfrac{1}{r}(1,-1,a)$ in order of increasing index r and increasing a within each index. So $Q_0 = \frac{1}{2}(1,1,1)$, $Q_1 = \frac{1}{3}(1,-1,1)$, etc.. The program *totalsearch.c* takes 2 integer arguments l and u, and searches through all baskets $\{n_i \times Q_i\}$ such that $l \leq \sum_{i=0}^{\infty} n_i(n_i+2) < u$.

III.9.12 The raw list.

Here is the first part of the list produced by the search program (with arguments 0 8).

X_6 in $\mathbb{P}(1, 1, 1, 1, 3)$
X_{12} in $\mathbb{P}(1, 1, 1, 4, 6)$
X_4 in $\mathbb{P}(1, 1, 1, 1, 1)$
X_5 in $\mathbb{P}(1, 1, 1, 1, 2)$
X_8 in $\mathbb{P}(1, 1, 1, 2, 4)$
X_{10} in $\mathbb{P}(1, 1, 1, 3, 5)$
$X_{2,3}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1)$
$X_{3,3}$ in $\mathbb{P}(1, 1, 1, 1, 1, 2)$
$X_{3,4}$ in $\mathbb{P}(1, 1, 1, 1, 2, 2)$
X_6 in $\mathbb{P}(1, 1, 1, 2, 2)$
$X_{4,4}$ in $\mathbb{P}(1, 1, 1, 1, 2, 3)$
X_7 in $\mathbb{P}(1, 1, 1, 2, 3)$
X_9 in $\mathbb{P}(1, 1, 1, 3, 4)$
$X_{2,2,2}$ in $\mathbb{P}(1, 1, 1, 1, 1, 1, 1)$
$X_{6,6}$ in $\mathbb{P}(1, 1, 1, 2, 3, 3)$
X_{12} in $\mathbb{P}(1, 1, 2, 3, 4)$
$X_{4,4}$ in $\mathbb{P}(1, 1, 1, 2, 2, 2)$
X_{10} in $\mathbb{P}(1, 1, 2, 2, 5)$
$X_{4,5}$ in $\mathbb{P}(1, 1, 1, 2, 2, 3)$
X_{18} in $\mathbb{P}(1, 1, 2, 6, 9)$
$X_{4,6}$ in $\mathbb{P}(1, 1, 1, 2, 3, 3)$
$X_{5,6}$ in $\mathbb{P}(1, 1, 1, 2, 3, 4)$
$X_{6,8}$ in $\mathbb{P}(1, 1, 1, 3, 4, 5)$

III.9.13 Refinement.

Of course this list contains complete intersections already obtained in other ways (see sections III.7 and III.8) and some intersections which do not meet the requirements; i.e.

(1) dimension 3,

(2) quasismooth but not the intersection of a linear cone with other hypersurfaces,

(3) well-formed,

(4) canonically or anti-canonically embedded,

(5) and have at worst terminal singularities.

The example $X_{6,22}$ in $\mathbb{P}(2, 2, 3, 4, 5, 11)$ from the raw list is not quasismooth, since the polynomial of degree 6 does not involve the generator of weight 5. We use the following lemma to cut out a large number of elements from the raw list produced by the search program.

III.9.14 Lemma: *Let $X_{d_1, \ldots, d_c}$ in $\mathbb{P}(a_0, \ldots, a_n)$ be quasi-smooth but not an intersection of a linear cone with other hypersurfaces. Suppose also that $d_1, \ldots, d_c$ and $a_0, \ldots, a_n$ are in increasing order. Then:*

(i) $d_c > a_n, d_{c-1} > a_{n-1}, \ldots, d_1 > a_{n-c+1}$.

(ii) *If* $d_{c-1} < a_n$ *then* $a_n | d_c$.

Proof. (*i*). Suppose $d_c > a_n, \ldots, d_{c-k+1} > a_{n-k+1}$ and $d_{c-k} < a_{n-k}$ for some $k = 0, \ldots, c-1$. So $d_i < a_{n-k}$ for all $i \leq c-k$. Therefore the polynomials $f_1, \ldots, f_{c-k}$ do not involve the variables $x_{n-k}, \ldots, x_n$. Let Π be the coordinate $(k+1)$-plane in $\mathbf{A}^{n+1}$ given by $x_0 = \ldots = x_{n-k-1} = 0$. So $f_1, \ldots, f_{c-k}$ are identically zero on Π. Define $Z = (f_{c-k+1} = \ldots = f_c = 0) \cap \Pi$. Thus $\dim Z \geq 1$ and so $Z - \underline{0}$ is nonempty. Let $Q \in Z - \underline{0}$. Then $\partial f_i / \partial x_j$ are zero at Q for all $i \leq c-k$ and for all j. Therefore

$$\operatorname{rank} \begin{pmatrix} \partial f_1/\partial x_0(Q) & \ldots & \partial f_1/\partial x_n(Q) \\ \vdots & & \vdots \\ \partial f_c/\partial x_0(Q) & \ldots & \partial f_c/\partial x_n(Q) \end{pmatrix} \leq k < c.$$

Thus $Q \in C_{\underline{f}}$ is singular and so X is not quasismooth.

(*ii*) is treated likewise.

□

III.9.15 Example. So a codimension 2 complete intersection X_{d_1, d_2} in $\mathbb{P}(a_1, \ldots, a_n)$, which is quasismooth and not the intersection of a linear cone with another hypersurface, satisfies:

(i) $d_2 > a_n$ and $d_1 > a_{n-1}$.

(ii) if $d_2 < a_n$ then $a_n | d_c$.

So this lemma give extra conditions to help remove *nasty* complete intersections.

The program was run between the limits 0 and 32 and gave the following list (after cutting out repetitions and nasty complete intersections):

Complete Intersection	K_X^3	p_g	Singularities.
$X_{2,2,2}$ in $\mathbb{P}(1,1,1,1,1,1,1)$	-8	0	
$X_{2,2,4}$ in $\mathbb{P}(1,1,1,1,1,1,1)$	16	7	
$X_{2,2,6}$ in $\mathbb{P}(1,1,1,1,1,1,3)$	8	6	
$X_{2,3,3}$ in $\mathbb{P}(1,1,1,1,1,1,1)$	18	7	
$X_{3,3,3}$ in $\mathbb{P}(1,1,1,1,1,1,2)$	27/2	6	$\frac{1}{2}(1,-1,1)$
$X_{3,3,4}$ in $\mathbb{P}(1,1,1,1,1,2,2)$	9	5	$2\times\frac{1}{2}(1,-1,1)$
$X_{3,4,4}$ in $\mathbb{P}(1,1,1,1,2,2,2)$	6	4	$4\times\frac{1}{2}(1,1,1)$
$X_{4,4,4}$ in $\mathbb{P}(1,1,1,1,2,2,3)$	16/3	4	$\frac{1}{3}(1,-1,1)$
$X_{4,4,4}$ in $\mathbb{P}(1,1,1,2,2,2,2)$	4	3	$8\times\frac{1}{2}(1,-1,1)$
$X_{4,4,5}$ in $\mathbb{P}(1,1,1,2,2,2,3)$	10/3	3	$\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{4,4,6}$ in $\mathbb{P}(1,1,1,2,2,3,3)$	8/3	3	$2\times\frac{1}{3}(1,1-,1)$
$X_{4,4,6}$ in $\mathbb{P}(1,1,2,2,2,2,3)$	2	2	$12\times\frac{1}{2}(1,1,1)$
$X_{4,5,6}$ in $\mathbb{P}(1,1,2,2,2,3,3)$	5/3	2	$2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$
$X_{4,6,6}$ in $\mathbb{P}(1,1,2,2,3,3,3)$	4/3	2	$4\times\frac{1}{3}(1,-1,1)$
$X_{4,6,6}$ in $\mathbb{P}(1,2,2,2,2,3,3)$	1	1	$18\times\frac{1}{2}(1,1,1)$
$X_{5,6,6}$ in $\mathbb{P}(1,1,2,2,3,3,4)$	5/4	2	$\frac{1}{4}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{5,6,6}$ in $\mathbb{P}(1,2,2,2,3,3,3)$	5/6	1	$4\times\frac{1}{3}(1,-1,1)$, $9\times\frac{1}{2}(1,1,1)$

Complete Intersection	K_X^3	p_g	Singularities.
$X_{6,6,10}$ in $\mathbb{P}(2,2,2,3,3,4,5)$	1/4	0	$\frac{1}{4}(1,-1,1)$, $22\times\frac{1}{2}(1,1,1)$
$X_{6,6,6}$ in $\mathbb{P}(1,2,2,2,3,3,4)$	3/4	1	$\frac{1}{4}(1,-1,1)$, $13\times\frac{1}{2}(1,1,1)$
$X_{6,6,6}$ in $\mathbb{P}(1,2,2,3,3,3,3)$	2/3	1	$8\times\frac{1}{3}(1,-1,1)$
$X_{6,6,6}$ in $\mathbb{P}(2,2,2,2,3,3,3)$	1/2	0	$27\times\frac{1}{2}(1,1,1)$
$X_{6,6,7}$ in $\mathbb{P}(1,2,2,3,3,3,4)$	7/12	1	$\frac{1}{4}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{6,6,8}$ in $\mathbb{P}(1,1,2,3,3,4,5)$	4/5	2	$\frac{1}{5}(1,-1,2)$
$X_{6,6,8}$ in $\mathbb{P}(1,2,2,3,3,4,4)$	1/2	1	$\frac{1}{4}(1,-1,1)$, $8\times\frac{1}{2}(1,1,1)$
$X_{6,6,8}$ in $\mathbb{P}(2,2,2,3,3,3,4)$	1/3	0	$18\times\frac{1}{2}(1,1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{6,7,8}$ in $\mathbb{P}(1,2,2,3,3,4,5)$	7/15	1	$\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$
$X_{6,8,10}$ in $\mathbb{P}(1,2,3,3,4,5,5)$	4/15	1	$2\times\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{6,8,10}$ in $\mathbb{P}(2,2,3,3,4,4,5)$	1/6	0	$2\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $14\times\frac{1}{2}(1,1,1)$
$X_{6,8,9}$ in $\mathbb{P}(1,2,3,3,4,4,5)$	3/10	1	$\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{2}(1,1,1)$
$X_{8,10,12}$ in $\mathbb{P}(2,3,4,4,5,5,6)$	1/15	0	$2\times\frac{1}{5}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $10\times\frac{1}{2}(1,1,1)$
$X_{8,9,10}$ in $\mathbb{P}(2,3,3,4,4,5,5)$	1/10	0	$2\times\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{4}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{9,10,12}$ in $\mathbb{P}(2,3,3,4,5,6,7)$	1/14	0	$\frac{1}{7}(1,-1,2)$, $6\times\frac{1}{3}(2,1,1)$, $5\times\frac{1}{2}(1,1,1)$
$X_{10,11,12}$ in $\mathbb{P}(2,3,4,5,5,6,7)$	11/210	0	$5\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,3)$
$X_{10,12,14}$ in $\mathbb{P}(2,3,4,5,6,7,8)$	1/24	0	$\frac{1}{8}(1,-1,3)$, $\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $8\times\frac{1}{2}(1,1,1)$
$X_{10,12,18}$ in $\mathbb{P}(3,4,5,5,6,7,9)$	2/105	0	$\frac{1}{7}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{12,14,15}$ in $\mathbb{P}(3,4,5,6,7,7,8)$	1/56	0	$\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{7}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$
$X_{12,15,16}$ in $\mathbb{P}(3,4,5,6,7,8,9)$	1/63	0	$2\times\frac{1}{2}(1,1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $\frac{1}{9}(1,-1,2)$
$X_{12,14,18}$ in $\mathbb{P}(4,5,6,6,7,8,9)$	1/105	0	$\frac{1}{7}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$

III.9.16 Note. After refinement there are no codimension 2 or 1 complete intersections left in the list.

III.9.17 Extra example. The family of intersections $X_{2,2,2,2,2}$ in $\mathbb{P}^8$ is smooth, $K_X^3 = 16$, $p_g = 9$ and $\chi(\mathcal{O}_X) = -8$.

If the search were continued this would eventually appear; however the program becomes painfully slow.

III.9.18 Conjectures.

(1) There are no canonical complete intersections with codimension greater than 5.

(2) There are no $\mathbb{Q}$-Fano complete intersections with codimension greater than 3.

III.9.19 K3 surfaces. Reid has done a similar search to produce lists of K3 surface weighted complete intersections; using Riemann-Roch for $\mathcal{O}_S(1)$ (see [R4, Theorem 9.1]). This time the search is finite due to the following theorem pointed out by Reid:

III.9.20 Theorem: *Let S be a K3 surface with canonical (Du Val) singularities of types A_{n_i}, D_{n_i} or E_{n_i} for $i = 1, \ldots, n$. So $\sum n_i \leq 19$. This limits the singularities present on the K3 surface to a finite list.*

Proof. Let $f : T \longrightarrow S$ be a minimal resolution. T is still a K3 surface. By [BP&V, Proposition VIII.3.3] $h^{1,1} = h^1(\Omega_T^1) = 20$. By the Signature Theorem [BP&V, Theorem IV.2.13] we have that the cup product restricted to $H^2(T, \mathbb{R})$ is non-degenerate of type $(1, h^{1,1}) = (1, 19)$. Via the Néron-Severi group, the exceptional (-2)-curves of the resolution f are linearly independent in $H^{1,1}$, each with negative self-intersection.

It is well known that a Du Val singularity of type A_n, D_n or E_n contributes exactly n (-2)-curves to T. Thus $\sum n_i \leq 19$.

$\square$

III.10 The search programs.

III.10.1 The language C++.

All the following programs are written in the C++ language (version 1.2), which is basically a superset of the C programming language (see [K&R]). [S] is a guide to C++. The programs were run on a VAX 11/750 with version 4.3 BSD UNIX.† C++ was used due to the existence of the user-defined type, called a *class*, which allows a lot of code to be hidden, simplifying the programs. For example the class 'rational' was defined as an triple of integers (n, p, q) corresponding to $n + p/q$, along with the corresponding functions of addition, subtraction, multiplication, etc., and with the concepts of comparison (equality, greater than, etc.). Other classes include 'complete intersection' (ie. $X_{d_1, \ldots, d_c}$ in $\mathbb{P}(a_0, \ldots, a_n)$), 'singularity' (which contains the type of a 3-fold cyclic quotient singularity $\frac{1}{r}(a, b, c)$) and 'record', which contains a record of pluridata (see Definition II.4.8).

III.10.2 rational.h.

```
/* Language: C++                                          */
/* Header file for the class definition of rationals      */

#include <stream.h>
#include <math.h>
```

† UNIX is a trademark of AT&T Bell Laboratories.

```
/* These functions must be overloaded before math.h                 */
/* and int.h are included.                                           */
overload floor;
overload ceil;
overload pow;

#include <math.h>
#include <int.h>

class rational {
        /* integer part, numerator and denominator respectively */
        int             whole, num, denom;

        void                    cancel();
        inline rational         inv();
public:
        rational()                      { whole = 0; num = 0; denom = 1;};
        rational(int n)                 { whole = n; num = 0; denom = 1;};
        rational(int p, int q)          { whole = 0; num = p; denom = q; cancel();};
        rational(int w, int p, int q)
                                        { whole = w; num = p; denom = q; cancel();};

        inline friend rational          operator+(rational, rational);
        inline friend rational          operator-(rational);
        inline friend rational          operator-(rational, rational);
        inline friend rational          operator*(rational, rational);
        inline friend rational          operator/(rational, rational);

        inline friend int               operator==(rational, rational);
        inline friend int               operator!=(rational, rational);
        inline friend int               operator>(rational, rational);
        inline friend int               operator<(rational, rational);
        inline friend int               operator>=(rational, rational);
        inline friend int               operator<=(rational, rational);

        inline friend int               numerator(rational);
        inline friend int               denominator(rational);
        inline friend int               whole(rational);
        inline friend int               remainder(rational);
        friend int                      floor(rational);
        friend int                      ceil(rational);
        friend rational                 pow(rational&, int);

        inline void                     operator+=(rational);
        inline void                     operator-=(rational);
        inline void                     operator*=(rational);
        inline void                     operator/=(rational);
        inline void                     operator++();
        inline void                     operator--();

        friend ostream                  &operator<<(ostream&, rational);
        friend istream                  &operator>>(istream&, rational&);
};
```

III.10.3 wp_space.h.

```
/* Language: C++                                                     */
/* Header file for the class definition of weighted                  */
/* projective spaces.                                                */

#include <stream.h>
#include <errno.h>

class wp_space
{
```

```
		int		dim;
		int*		weight;
public:
		~wp_space ();
		wp_space ();
		wp_space (int);
		wp_space (wp_space&);
		wp_space (int, int*);

		void				operator=(wp_space&);
		inline friend int		dimension(wp_space& P);
		inline friend int*		weight(wp_space& P);
		inline friend int		operator!=(wp_space&, wp_space&);
		friend int			operator==(wp_space&, wp_space&);
		friend int			dimH(wp_space&, int);
		friend int			mod(wp_space& P);
		friend int			prod(wp_space& P);
		friend ostream&			operator<<(ostream& s,wp_space& P);
		friend istream&			operator>>(istream& s,wp_space& P);
};
```

III.10.4 c_int.h.

```
/* Language: C++                                          */
/* Header file for the class definition of                */
/* complete intersections.                                */

overload dimension;
overload weight;
#include <wp_space.h>
#include <rational.h>

class c_int
{
		int			codim;
		int*			degree;
		wp_space		ambP;

		void cancel();
public:
		~c_int ();
		c_int (int, int* , wp_space&);
		c_int (int, int* , int, int*);
		c_int (c_int&);
		c_int ();
		c_int (int d);
		c_int (wp_space&);

		void				operator=(c_int&);
		inline friend int		dimension(c_int C);
		inline friend int		codimension(c_int C);
		inline friend int		pg(c_int C);
		inline friend int*		degree(c_int& C);
		inline friend int*		weight(c_int& C);
		inline friend wp_space		ambient(c_int C);
		inline friend int		operator!=(c_int&, c_int&);
		friend int			operator==(c_int&, c_int&);
		friend int			amplitude(c_int& C);
		friend rational			K(c_int& C);
		friend ostream&			operator<<(ostream& s,c_int& ambP);
		friend istream&			operator>>(istream& s,c_int& ambP);
};
```

III.10.5 singularity.h.

```
/* Language: C++                                                                  */
/* Header file for singularities                                                  */
#include <stream.h>
#include <errno.h>

class singularity
{
        int             index;
        int             a[3];
public:
        singularity ()          {index=1; a[0]=1; a[1]=-1; a[2]=1;};
        singularity (int r, int aa, int bb, int cc)
                                {index=r; a[0]=aa; a[1]=bb; a[2]=cc;};

        inline friend int               index(singularity);
        inline friend int*              a(singularity);
        inline friend int               operator!=(singularity, singularity);
        friend int                      operator==(singularity, singularity);
        friend int                      is_isolated(singularity);
        friend int                      is_canonical(singularity);
        friend int                      is_terminal(singularity);
        friend singularity              standard(singularity);
        friend singularity              next_sing(singularity);
        inline friend ostream           &operator<<(ostream&, singularity&);
        friend istream&                 operator>>(istream&, singularity&);
};
```

III.10.6 record.h.

```
/* Language: C++                                                                  */
/* Header file for records (of pluridata).                                        */
#include <rational.h>
#include <singularity.h>

#define NO_TYPES 200

struct record
{
        rational        K3;
        int             chi,pg;
        int             no_types;       /* no. of types of sing in list S
        singularity     S[NO_TYPES];
        int             n[NO_TYPES];    /* no. of sing. of type S[i].
};

/* All the function definitions.                                                  */

rational        l(singularity, int);
rational        delta_l(singularity, int);
int             Pgenus(record&, int);
int             delta_P(record&, int);
ostream&        operator<<(ostream&, record&);
istream&        operator>>(istream&, record&);
```

III.10.7 hyp.c.

The following program 'hyp.c' is used to search for 3-fold weighted hypersurfaces with only isolated quotient terminal singularities. Versions of this program were used to search for surfaces and curves. The technique is very basic, scanning through all the 5-tuples $(a_0, ..., a_4)$ such that:

$$\text{lower} < a_0 + ... + a_4 < \text{upper}.$$

If the executable code is in 'hyp' then it is called via:

hyp upper lower amplitude

where $K_X \equiv O_X$(amplitude). For example:

hyp 4 100 1

will produce all the canonically embedded 3-fold hypersurfaces (i.e. the table in III.7.1 but without K_X^3, p_g and the singularities) with

$$4 < a_0 + \ldots + a_4 < 100.$$

Clearly this is not a complete search.

III.10.8 The source code.

```
/* Language: C++                          Program: hyp.c                */
/* Program to search for 3-fold weighted hypersurfaces                  */
/* in order of increasing suma = a[0] + ... + a[4]                      */
/* The amplitude must be specified.                                     */

#include <stream.h>
#define PASS 0

int     hcf(int a, int b);          /* returns hcf of 2 integers
int     test1(int a[], int d);      /* tests for non-canonical sings
void    print(int a[], int d);      /* prints out any hypersurface found
void    exit(int);                  /* exits from program

main(int argc, char *argv[])                                              main
{
        int     lower, upper, amplitude;
        int     d, a[6];            /* Represents X(d) in P(a).
        int     num = 0;
        int     suma, s0, s1, s2;

        /* Checking for correct use of this routine                     */
        if (argc != 4)
        {
                cerr<< "Usage: " << argv[0] << " lower upper amplitude\n";
                exit(1);
        }

        /* Input of limits and amplitude                                */
        sscanf(argv[1], "%d", &lower);
        sscanf(argv[2], "%d", &upper);
        sscanf(argv[3], "%d", &amplitude);

        for (suma = lower; suma < upper; ++suma)
        {
                /* Main searching loop */
                for (a[0] = 1; a[0] <= suma/5; ++ a[0])
                {
                 s0 = suma - a[0];
                 for (a[1] = a[0]; a[1] <= s0/4; ++a[1])
                 {
                  s1 = s0 - a[1];
                  for (a[2] = a[1]; a[2] <= s1/3; ++a[2])
                  {
                   s2 = s1 - a[2];
                   for (a[3] = a[2]; a[3] <= s2/2, ++a[3])
                   {
                    a[4] = s2 - a[3];
                    if (a[4] >= a[3])
                    {
                            d = suma + amplitude;
```

```
                        /* Testing for X(d) in P(a) to have                         */
                        /* only terminal singularities.                             */
                        if (test1(a, d) == PASS)
                        {
                                /* Output of hypersurface X(d) in P(a). */
                                print(a, d);
                        }
                    }
                  }
                 }
                }
               }
        }
}

/* Output of hypersurface X(d) in P(a)                                          */
void print(int a[5], int d)
{
        cout<< "X(" << d << ") in ";
        cout<< "P(" << a[0] << ", " << a[1];
        cout<< ", " << a[2] << ", " << a[3];
        cout<< ", " << a[4] << ")\n";
}

/* This tests hypersurfaces in weighted projective                              */
/* spaces to have only isolated terminal singularities                          */

test1(int a[], int d)                                                          test1
{
        int         h2, i, j, m;
        int         e[5];
        int         amplitude = d - a[0] - a[1] - a[2] - a[3] - a[4];

        if (d <= a[4] || d <= 0)
                    return 1;

        for (i = 0; i < 5; ++i)
        {
                    if (d % a[i] != 0)
                    {
                                /* Testing for terminal sing at the vertices          */
                                for (j = 0; j < 5 && (d - a[j]) % a[i] != 0; ++j)
                                            ;
                                if (j == 5)
                                            return 2;
                                e[i] = j;

                                for (m = 0; m < 5 && (m == i || m == e[i] || (a[m] + amplitude) % a[i] != 0); ++m)
                                            ;
                                if (m == 5)
                                            return 3;
                    }
                    for (j = i + 1; j < 5; ++j)
                    {
                                h2 = hcf(a[i], a[j]);
                                if (h2 != 1)
                                {
                                            /* Testing for terminal sings along PiPj */
                                            if (d % h2 != 0)
                                                        return 4;

                                            for (m = 0; m < 5 && (m == i || m == j || (a[m] + amplitude) % h2 != 0); ++m)
                                                        ;
                                            if (m == 5)
                                                        return 5;
```

```
                                        for (m = j + 1; m < 5; ++m)
                                                if (hcf(h2, a[m]) != 1)
                                                        return 0;
                                }
                        }
                }
                return PASS;
}
```

III.10.9 Totalsearch.

This program searches through every combination of baskets of singularities and P to find (anti-) canonically embedded 3-folds (see section III.9.11).

main

```
/* Language: C++                    Program: totalsearch.c                    */
/* Program to seach through each combination of P and basket                  */
/* of singularities.                                                          */

#include <record.h>
#include <c_int.h>

/* Sizes of the table.                                                        */
#define MAXCOL 50
#define MAXROW 51

#define TRUE 1
#define FALSE 0
#define ON 1
#define OFF 0

void            test(record&, int, int, int);
void            totalsum(record&, int);
c_int           tablesum(int p[], int);

main(int argc, char *argv[])
{
                int             lower, upper;
                int             i, ntype, level, sum;
                int             fin, end, Rflag = OFF;
                int             n = 1, P, maxnp = 0;
                record          R;
                rational p, l2[NO_TYPES];

        if (argc < 3 || argc > 5)
        {
            cerr << "Usage: " << argv[0] << " [-R] lower upper [maxnp]\n";
            exit(1);
        }

                if (argv[1][0] == '-')
                {
                        if (argv[1][1] == 'R')
                        {
                                Rflag = ON;
                        }
                        else
                        {
                                cerr<< "Usage: " << argv[0] << " can only take the flag -R\n";
                                exit(1);
                        }
                        n = 2;
                }

        sscanf(argv[n], "%d", &lower);
        sscanf(argv[n+1], "%d", &upper);
```

```
if (argc == n+3)
{
		sscanf(argv[n+2], "%d", &maxsop);
}

/* Initialising singularity list.                                  */
R.S[0] = singularity(2,1,1,1);
l2[0] = l(R.S[0], 2);

for (ntype=1; ntype<lower; ++ntype)
{
		R.S[ntype] = next_sing(R.S[ntype-1]);
		l2[ntype] = l(R.S[ntype], 2);
}

/* Starting main loop.                                             */
for (ntype=lower; ntype<upper; ++ntype)
{
		/* Initialisation for search 'loop'                          */
		sum = 0;
		fin = FALSE;
		for (i=0; i < ntype; ++i)
				R.n[i] = 0;

		/* Start 'loop'                                              */
		while (fin == FALSE)
		{
				P = ntype-sum;
				/* Calc. of l(2)                                           */
				p = 0;
				for (i=0; i<ntype; ++i)
				{
						p += R.n[i]*l2[i];
				}

				R.no_types = ntype;

				/* Q-Fano                                                  */
				R.K3 = 2*(3 - p - P);
				if (R.K3 < 0)
				{
						R.pg = 0;
						R.chi = 1;
						test(R, P, maxsop, Rflag);
				}

				/* Canonical                                               */
				for (R.pg = 0; 3*R.pg < 3-p+P; ++R.pg)
				{

						R.K3 = 2*(3 - p + P - 3*R.pg);
						if (R.K3 > 0)
						{
								R.chi = 1 - R.pg;
								test(R, P, maxsop, Rflag);
						}
				}
				/* End of search 'loop'; testing and update                   */
				level = 0;
				end = FALSE;
				while (end == FALSE && level < ntype)
				{
						if (sum+level+1 > ntype)
						{
								sum -= (level+1)*R.n[level];
```

```
                        ++level;
                }
                else
                {
                        sum += level+1;
                        ++R.n[level];
                        for (i=0; i<level; ++i)
                                R.n[i]=0;
                        end = TRUE;
                }
            }
            if (level >= ntype)
                fin = TRUE;
        }
        /* Next singularity                                                          */
        if (ntype != 0)
        {
            R.S[ntype] = next_sing(R.S[ntype-1]);
            l2[ntype] = l(R.S[ntype], 2);
        }
    }
}

void test(record& R, int P, int maxop, int Rflag)
{
    if (Rflag == ON)
    {
        cout<< "P = " << P << ", " << R;
    }
    totaltest(R, maxop);
}

/* This calculates the (anti) plurigenera and sends them off        */
/* to the routine that calculated the table.                        */

void totaltest(record& R, int maxop)
{
    int     p[MAXROW];
    int     i;
    c_int   C;
    c_int   PP = up_space(0);

    p[0] = 1;
    if (R.K3 > 0)
    {
        p[1] = R.pg;
        p[2] = Pgenus(R,2);             /* returns the P2 of R.
        if (p[2] < 0)
            return;

        /* Calculates the plurigenera corresponding                          */
        /* to the record R.                                                  */
        for (i=3; i< MAXROW; ++i)
        {
            p[i] = Pgenus(R,i);
        }
    }
    else
    {
        if (R.K3 < 0)
        {
            p[1] = Pgenus(R, -1);
            if (p[1] < 0)
                return;
```

```
                        /* Calculates the anti-plurigenera                              */
                        /* corresponding to the record R.                               */
                        for (i=2; i< MAXROW; ++i)
                        {
                                p[i] = Pgenus(R, -i);
                        }
                }
                else
                {
                        /* Error message if R.K3 == 0                                   */
                        cout<< "totaltest: Record: " << R;
                        cout<< "K numerically trivial.\n";
                        return;
                }
        }

        /* Tests the list of (anti) plurigenera                         */
        /* Returns the value of PP if it fails.                         */
        C = tabletest(p, maxcp);

        if (C != PP)
        {
                cout<< C << "\n";
                cout.flush();
        }

        return;
}

/* Table to determine possible complete intersection           */
/* for given parameters                                         */
/* Returns the complete intersection if successful or          */
/* the value wp_space(0) if not.                                */

c_int tabletest(int p[], int maxcp)
{
        int     q[MAXCOL][MAXROW], a[MAXCOL];
        int     deg[MAXCOL], wght[MAXCOL];
        int     col = 0, row = 1;
        int     gens = 0, value = 0;
        int     i, j;

        /* Initialise table.                                            */
        q[0][0] = 1;
        q[0][1] = p[1];

        /* Extend table and test for complete intersection.            */
        while (row < MAXROW-1 && col < MAXCOL-1)
        {
                /* Extend table.                                                */
                if (q[col][row] == 0)
                {
                        /* next row.                                                    */
                        ++row;
                        q[0][row] = p[row];
                        for (i=0; i<col; ++i)
                        {
                                if (a[i]>0)
                                        q[i+1][row] = q[i][row]-q[i][row-a[i]];
                                else
                                        q[i+1][row] = q[i][row]+q[i+1][row+a[i]];
                        }
                }
                else
                {
```

```
                        /* Next column.                                              */
                        ++col;

                        q[col][0] = 1;
                        for (i=1; i<row; ++i)
                                q[col][i] = 0;

                        /* Test for generator.                                       */
                        if (q[col-1][row]>0)
                        {
                                a[col-1] = row;
                                q[col][row] = q[col-1][row]-1;
                                ++gens;
                        }
                        else
                        {
                                a[col-1] = -row;
                                q[col][row] = q[col-1][row]+1;
                                ++relns;
                        }
                }
        }

        if (col == MAXCOL - 1)
                return wp_space(0);

        /* Output of table.                                  */
        if (maxop != 0)
        {
                cout<< "\n\n";
                for (i=0; i<col; ++i)
                        cout<< "\t(" << a[i] << ")";
                cout<< "\n";

                for (i=0; i<row && i< maxop; ++i)
                {
                        cout<< i;
                        for (j=0; j<=col; ++j)
                        {
                                cout<< "\t" << q[j][i];
                        }
                        cout<< "\n";
                }
        }

        /* Separation of degrees and weights.                */
        for (i=0, j=0; i<relns; ++i)
        {
                while (a[j]>0)
                        ++j;
                deg[i] = - a[j];
                ++j;
        }
        for (i=0, j=0; i<gens; ++i)
        {
                while (a[j]<0)
                        ++j;
                wght[i] = a[j];
                ++j;
        }

        return c_list(relns, deg, gens-1, wght);
}
```

IV

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